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There are three routes to wisdom:

1. *reflection*, this is the noblest,
2. *imitation*, this is the easiest,
3. *experience*, this is the bitterest.

CONFUCIUS

To my family.

Contents

Introduction

The subject of this thesis is placed at the interface between the theory of inequalities and approximation theory. Chebyshev-, Grüss- and Ostrowski-type inequalities have attracted much attention over the years, because of their applications in mathematical statistics, econometrics and actuarial mathematics.

The classical form of Grüss' inequality, first published by G. Grüss in [62], gives an estimate of the difference between the integral of the product and the product of the integrals of two functions in $C[a,b]$. In the succesive years, many variants of this inequality appeared in the literature.

The aim of this thesis is to clarify the terminology that was not exactly presented in a transparent way until now, to remember well-known Chebyshev-Grüss- and Ostrowski-type inequalities that have already been studied and to introduce new results, in both the univariate and bivariate case. These results can then be generalized to the multivariate case, but this remains to be studied in the future. We also want to point out that all of the inequalities of Chebyshev-Grüss-type given here are for two or more functions of the same type.

When considering the classical Grüss inequality, we observe that on the left-hand side of the estimate is the well known classical Chebyshev functional [25], while the right-hand side is of Grüss-type, i.e., it includes differences of upper and lower bounds of the two functions in question. The Grüss inequality for the Chebyshev functional explains the non-multiplicativity of the integration. In our research, we are interested in how non-multiplicative can a linear functional in the worst case be. In order to give an answer to this question, we consider the generalized Chebyshev functional

$$T_L(f,g) := L(f \cdot g) - L(f) \cdot L(g),$$

for a positive linear functional L, and use the terminology "Chebyshev-Grüss-type inequalities", when we talk about Grüss inequalities for special cases of generalized Chebyshev functionals. We therefore obtain a general form of such estimates,

$$|T_L(f,g)| \leq E(L,f,g),$$

where the right-hand side is an expression depending on different properties of L and some kind of oscillations of the functions in question.

Another renowned classical inequality was introduced by A. M. Ostrowski in [89] and can be given in a variety of forms. The Ostrowski-type inequalities we recall and introduce all give different upper bounds for the approximation of the average value by a single value of the function in question. The approach considered by A. M. Ostrowski and a lot of other investigations on the topic were carried out assuming differentiability properties of the functions. In comparison, A. Acu and H. Gonska [2] show that such conditions are not necessary and give a generalization of Ostrowski's inequality for an arbitrary continuous function $f \in C[a,b]$ and certain linear operators. On the right-hand side the least concave majorant of the modulus of continuity of the given function is used. Thus the upper bounds of our general form of Ostrowski's inequalities will involve least concave majorants, functions from certain classes, their norm or an expression derived from the functions in question.

Both cases have been intensively investigated by S.S. Dragomir and other authors (see Chapters $2-9$ in the recent monography of G. Anastassiou [9], Ch. XV on "Integral inequalities involving functions with bounded derivatives" in the book by D. S. Mitrinović et al. [84]; the reader can also consult the books [85], [22] and the references therein).

In the present thesis we continue to consider the recent method in which the functional L is obtained by composing a point evaluation functional with a positive linear operator in both the Chebyshev-Grüss and the Ostrowski process.

The first approach in this direction was made in a paper by A. Acu, H. Gonska and I. Raşa [2] from 2011 and is extended in the present thesis. One essential feature of it is the systematic use of the least concave majorant of the first order modulus of continuity which first appeared in this context in a paper by B. Gavrea and I. Gavrea [40] (mostly in the Ostrowski case). The use of the concave majorant has the advantage, that the deviation in the Chebyshev-type functional is also measured for all continuous functions on a compact metric space and not only for those having certain regularity properties, such as satisfying a Lipschitz condition with exponent 1. Such inequalities are obtained via the use of a suitable $K-$ functional (see the paper of R. Păltănea [91]).

In all our estimates for the Chebyshev-Grüss-type inequalities the second moments of the positive linear operators in question or a closely related quantity play an essential role. That these two quantities may lead to different upper bounds will be shown by the use of several interpolation operators which do not reproduce linear functions. We consider both the cases of a compact interval $[a, b]$ and the half-open semiaxis $[0, \infty)$.

A second approach considered in the present work is that of Chebyshev-Grüss inequalities via discrete oscillations. The latter indeed competes with that via the concave majorant in that there are situations in which the first is better than the second and vice versa. All the results are applied to a variety of well-known positive linear operators in both the univariate and the bivariate cases. In contrast to that, we show that also for certain non-positive Lagrange operators similar results can be obtained which, however, are of a less elegant form.

The Ostrowski-type inequalities given in the end of the thesis complete our presentation in the spirit of the many papers dealing with both types of inequalities. Our results given there build up a short parallel of the ones given in the Chebyshev-Grüss case. The applications included are not only for special positive linear operators, but also for their iterates and for differences of such operators. Some of these applications are extended to the bivariate case.

The thesis consists of five chapters.

The **first chapter** comprises preliminary instruments that will be further used for deriving our results. This thesis is based upon some main tools: the moduli of smoothness, the $K-$ functional and its connection to the moduli, positive and not-necessarily positive linear operators. The moduli are given in two different settings, i.e., for functions defined both on compact intervals of the real axis and on a compact metric space, in the univariate case. $K-$ functionals and the way they are connected to the moduli are given in both frames. For the bivariate case, that will be also treated later on, we are only interested in the moduli of continuity of functions defined on the product of two metric spaces (see Subsection 1.1.5).

Section 1.2 recalls many positive linear operators that have intensively been studied in the literature. All of them reproduce constant functions, some of them do not reproduce linear functions and this last property will be an advantage in order to improve some inequalities.

The operators that we illustrate here represent an interesting variety. Most of them are defined on compact intervals, but we also consider operators for functions defined on infinite intervals. We discuss the well-known Bernstein operators but also some interesting generalizations. The BLaC operators give an exotic touch to the survey. In the end of the chapter, the Lagrange interpolation operator is also studied, in order to see what happens if positivity is not taken into consideration.

In the **second chapter** we talk about Chebyshev-Grüss-type inequalities in the univariate case. First some auxiliary and historical results are given, in the two settings that were mentioned before. These results are recalled in order to motivate our research and because some of them will be slightly improved. Applications of the auxiliary results involving some positive linear operators are reviewed. Some remarks and results concerning Chebyshev's inequality are also presented. We give another proof for a Chebyshev-Grüss-type inequality involving a positive linear functional L, an inequality that was proven in another way in [2]. We then introduce (pre-)Chebyshev-Grüss-type estimates in both settings, using second moments, first absolute moments and quantities involving differences of second and first moments. We then apply the main results to the (positive) linear operators discussed in the first chapter. For these applications, oscillations expressed by the least concave majorant of the first order modulus are used in the first place. The use of such oscillations includes all points in the considered intervals, and this is the reason why a new approach involving less points arises. The discrete oscillations defined in Subsection 2.2.6 represent the grounds upon which this approach was constructed. The discrete linear functional case is introduced and applied to the Lagrange operators. In case of positivity, we apply the discrete positive linear functional case to some positive linear operators. Of great interest here are the sums of squares of the fundamental functions of the operators, which need to be minimized. Due to this new approach, we can also give Chebyshev-Grüss-type inequalities for operators defined for functions given on infinite intervals (see Subsections 2.2.7.4, 2.2.7.5 and 2.2.7.6). When talking about discrete oscillations, we give Chebyshev-Grüss-type inequalities for more than two functions at the end of this chapter. This is motivated by the last section of article [2], where the authors introduced an inequality on a compact metric space for more than two functions, using the least concave majorant. We compare our result to theirs.

The **third chapter** extends the results from the univariate to the bivariate case. We use the method of parametric extensions involving the product of two compact metric spaces. Auxiliary and historical results are also recalled in the first part. We then choose some of the operators presented in the beginning and construct their tensor products. For these operators we also define the first, second and first absolute moments, which we will need for our main results in Section 3.3. The applications are given for both the approach with the least concave majorant and the one via discrete oscillations.

The purpose of the **fourth and fifth chapters** is to complete this work, in the sense that we also consider univariate and bivariate Ostrowski-type inequalities. In Sections 4.1 and 5.1 we again recall some historical results, inequalities that were further studied and modified. In Section 4.2 we give a result that modifies in some sense the inequality given by A. Acu and H. Gonska in [1]. Some additional results are given in the form of corollaries. Corollary 4.2.4 is applied to iterates of different positive linear operators. Moreover, Corollary 4.2.2 is also applied in the case of differences of positive linear operators, as can be seen in Section 4.4. The last chapter introduces two examples of Ostrowski-type inequalities in the bivariate case. The two applications given here are for products of Bernstein-Stancu and Bernstein-Durrmeyer operators with Jacobi weights. In both cases, we get Ostrowski-type inequalities with or without involving the iterates of the operators. The limit of the iterates of the positive linear operators is also investigated.

There is a connection between the Ostrowski and the Grüss inequalities, which explains the term "Ostrowski-Grüss-type inequalities" that was often used in the literature. For clarity, we emphasize that we exclusively use the term when the lower bound is the error term in a rather simple quadrature formula (like in Ostrowski's article [89]), while the upper bound contains differences of bounds as used in the paper by G. Grüss [62]. In order to complete the historical remarks, the following reminders appear to be in order. It seems that the term "Ostrowski-Grüss-type inequality" was coined by Dragomir et al. in [33]. The term also appeared in a paper by Cerone et al. (see [23]). A more substantial paper from 2000 using the term is one by Matić et al. (see [79]). For more details, the reader should consult the papers [55], [56], [3] and the references therein.

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Notations and symbols

In this work we shall often make use of the following symbols:

:=	is the sign indicating equal by definition".
	a:=b" indicates that a is the quantity to be defined or explained,
	and b provides the definition or explanation. b=:a" has
	the same meaning.
$\mathbb{N}$	the set of natural numbers,
$\mathbb{N}_0$	the set of natural numbers including zero,
$\mathbb{R}$	the set of real numbers,
$\mathbb{R}_+$	the set of positive real numbers,
$[a,b]$	a closed interval,
(a,b)	an open interval.
	Let X be an interval of the real axis.
$B(X)$	the set of all real-valued and *bounded* functions defined on X.
$L^p(X)$	the class of the *p-Lebesgue integrable* functions on $X,\ p \geq 1$.
$\|f\|_p$	is the norm on $L^p(X)$ defined by $\|f\|_p := \left(\int_X \lvert f(x)\rvert dx\right)^{1/p},\ p \geq 1$.
$C(X)$	the set of all real-valued and *continuous* functions defined on X.
$C_b(X)$	the set of all real-valued functions, defined by $C_b(X) := C(X) \cap B(X)$.
$C[a,b]$	the set of all real-valued and *continuous* functions defined on the compact
	interval $[a,b]$.
	For $f \in B(X)$ or $f \in C(X)$
$(X,d),\ (X,d_X)$	Metric spaces equipped with metric d (or d_X).
$d(X)$	Diameter of the compact metric space (X,d).
$\lvert f\rvert_\infty$	is the *Chebyshev norm* or *sup-norm*, namely
	$\|f\|_\infty := \sup\{\lvert f(x)\rvert : x \in X\}$.
$C^r[a,b]$	the set of all real-valued, *r-times continuously differentiable* function,
	$(r \in \mathbb{N})$.
$\mathrm{Lip}_r M$	the set of all $C[a,b]-$ functions that verify the *Lipschitz condition*:
	$\lvert f(x_2) - f(x_1)\rvert \leq M\lvert x_2 - x_1\rvert^r,\ \forall x_1, x_2 \in [a,b], 0 < r \leq 1,\ M > 0$.
$\prod_n$	$(\prod_n[a,b],\ n \in \mathbb{N}_0)$ the linear space of all real polynomials
	with the degree at most n.
1_X	$\mathbb{R} \ni x \mapsto 1 \in \mathbb{R}$, $X \neq \emptyset$ an arbitrary set .
$f_X,\ f^Y$	Partial functions of bi-or multivariate functions.
e_n	denotes the $n-$th *monomial* with $e_n : [a,b] \ni x \mapsto x^n \in \mathbb{R}, n \in \mathbb{N}_0$.
	For a function $f : X \to \mathbb{R},\ X$ an interval of the real axis we have:
ω...	exclusively used to denote moduli of smoothness of various kinds.
$\omega_d(f,t)$	(Metric) modulus of continuity, defined for functions $f \in C(X)$,
	(X,d) a compact metric space, $t \geq 0$.
$\widetilde{\omega_d},\ \tilde{\omega}$	Least concave majorant of a metric modulus of continuity.
$\omega_1(f;t)$	(univariate) 1-st order modulus of smoothness, defined using

$\Delta_h f(x)$	Difference of order 1 with increment h and starting point x.
$\omega_k(f;t_1,t_2)$	(Bivariate) total modulus of smoothness of order k, defined for functions $f \in C(X)$, $X \subset \mathbb{R}^2$ compact, and $t_i \geq 0$, $i = 1,2$.
D^r or $f^{(r)}$	$r-$th *derivative* of the function $f \in C^r[a,b]$.
$[x_0,\ldots,x_m;f]$	$m-$th divided difference of $f \in \mathcal{F}(X)$ on the not necessarily distinct knots $x_0,\ldots,x_m \in X$.
$a^{\overline{b}}$	are the *rising factorials* $a^{\overline{b}} := \prod_{i=0}^{b-1}(a+i),\ a \in \mathbb{R},\ b \in \mathbb{N}_0$, where $\prod_{i=0}^{-1} := 1$.
$a^{\underline{b}}$	are the *falling factorials* $a^{\underline{b}} := \prod_{i=0}^{b-1}(a-i),\ a \in \mathbb{R},\ b \in \mathbb{N}_0$, where $\prod_{i=0}^{-1} := 1$.
$y^{[m,h]}$	the *factorial power* of step $h \in \mathbb{R}$ defined by: $y^{[m,h]} := \prod_{i=0}^{m-1}(y-ih)$, $m \in \mathbb{N}_0$. As above $\prod_{i=0}^{-1} := 1$.
$(X, \Vert\cdot\Vert_X)$,	Function space X equipped with the norm $\Vert\cdot\Vert_X$
$((X, \vert\cdot\vert_X))$	(the seminorm $\vert\cdot\vert_X$).
$B(X), B_{\mathbb{R}}(X)$	Space of all real-valued and bounded functions om the set $X \neq \emptyset$.
$C(X), C_{\mathbb{R}}(X)$	Space of all real-valued and continuous functions on the topological space X.
Lip_r	Space of Lipschitz continuous functions (with exponent r).
$C^r, C^r(I)$ and $C^r[a,b]$	Space of all real-valued functions on $I = [a,b]$ having continuous derivatives up to order r.
$\Vert\cdot\Vert$	If not otherwise indicated, denotes the Chebyshev (max, sup) norm.
$\Vert\cdot\Vert_\infty$, $\Vert\cdot\Vert_X$	Sometimes used to denote the Chebyshev norm, and the Chebyshev norm over the set X, respectively.
$\vert g\vert_{Lip_r}$	Lipschitz seminorm of a function $g \in C(X,d)$; smallest Lipschitz constant.
I, I_X, I_Y	Identity operator (canonical embedding) on a function space.
X_L, Y_L	Parametric extension of the univariate operator L.
$\Vert L\Vert_{[X,Y]}$, $\Vert L\Vert$	Canonical norm of an operator L, usually mapping a normed space X into a normed space Y; the second notation is used when it is clear what X and Y are.
$\mathcal{O}, o$	Landau notations.
$[x]$	the integral part of a real number x (i.e., the greatest integral number, that doens't exceed x).
$Supp(f)$	Support of a function f.
$Supp(\mu)$	Support of a measure μ.

1 Preliminaries

1.1 Moduli of smoothness and K-functionals

The moduli of smoothness (continuity) and K-functionals, used in connection to the moduli, will be of interest in the whole thesis. We recall definitions and properties of these moduli for real-valued and continuous functions defined both on a compact metric space (X, d) and on a compact interval $[a, b], a < b$, of the real axis. We will extend the results obtained in the compact metric space to the bivariate case.

1.1.1 Moduli of univariate functions defined on compact intervals of the real axis

When we want to establish the degree of convergence of positive linear operators towards the identity operator, we use first kind moduli of smoothness.

Definition 1.1.1. For a function $f \in C[a, b]$ and $t \geq 0$, we have

$$\omega_1(f; t) := \sup\{|f(x+h) - f(x)| : x, x+h \in [a, b], 0 \leq h \leq t\}.$$

Above we gave one definition of the first moduli of smoothness. This was presented in the Ph. D. thesis of D. Jackson [65]. The name of the modulus comes from the following proposition.

Proposition 1.1.2. *Let $f \in C[a, b]$ and $t > 0$. Then the following properties hold:*

a) If $\lim\limits_{t \to 0^+} \omega_1(f; t) = 0$ then f is continuous on $[a, b]$.

b) The following equivalence is given: $f \in Lip_r M$ if and only if $\omega_1(f; t) \leq M \cdot t^r$, for $0 < r \leq 1$ and $M > 0$.

A very important tool that we use is the least concave majorant of the modulus of continuity $\omega_1(f; \cdot)$. This is given by

$$\begin{aligned} \widetilde{\omega_1}(f; t) &= \widetilde{\omega}(f; t) \\ &:= \begin{cases} \sup\limits_{0 \leq x \leq t \leq y \leq b-a, x \neq y} \frac{(t-x)\omega_1(f;y) + (y-t)\omega_1(f;x)}{y-x}, & \text{for } 0 \leq t \leq b-a, \\ \widetilde{\omega}(f; b-a) = \omega_1(f; b-a), & \text{if } t > b-a, \end{cases} \end{aligned} \tag{1.1.1}$$

from which we get a relationship between the different moduli:

$$\omega_1(f; \cdot) \leq \widetilde{\omega}(f; \cdot) \leq 2 \cdot \omega_1(f; \cdot).$$

For more properties of the moduli, including $\widetilde{\omega}(f; \cdot)$, see [45]. N.P. Korneičuk gave a proof in [70] for the relationship between the function $\omega(f; \cdot)$ and its least concave majorant $\widetilde{\omega}(f; \cdot)$

$$\widetilde{\omega}(f; \xi \cdot \varepsilon) \leq (1 + \xi) \cdot \omega(f; \varepsilon),$$

for any $\varepsilon \geq 0$ and $\xi > 0$. It was also showed that this inequality cannot be improved for each $\varepsilon > 0$ and $\xi = 1, 2, \ldots$.

1.1.2 K-functionals and the connection to the moduli

When we are interested in measuring the smoothness of functions, we can also use the so-called Peetre's K-functional. It was introduced, as the name suggests, by J. Peetre in 1968 [92] and can be defined in a very general setting. However, we need the following form in this thesis.

Definition 1.1.3. For any $f \in C[a,b]$, $t \geq 0$ and $s = 1$, we denote

$$\begin{aligned} K_{s=1}(f;t)_{[a,b]} &:= K(f;t;C[a,b],C^1[a,b]) \\ &:= \inf_{g \in C^1[a,b]} \{\|f-g\|_\infty + t \cdot \|g'\|_\infty\} \end{aligned}$$

to be Peetre's K-functional of order 1.

We first recall some properties of the above K-functional, proven by P.L. Butzer and H. Berens for $s \geq 1$ (see [19]). For other references, see [32] and [102].

Lemma 1.1.4. *(see Proposition 3.2.3 in [19])*

(i) The mapping $K_1(f;t) : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous especially at $t = 0$, i.e.,

$$\lim_{t \to 0^+} K_1(f;t) = 0 = K_1(f;0).$$

(ii) For each fixed $f \in C[a,b]$, the application $K_1(f;\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is monotonically increasing and a concave function.

(iii) For arbitrary $t_1, t_2 \geq 0$ and $f \in C[a,b]$, one has the inequality

$$K_1(f;t_1 \cdot t_2) \leq max\{1, t_1\} \cdot K_1(f;t_2).$$

(iv) For arbitrary $f_1, f_2 \in C[a,b]$ and $t \geq 0$, we have

$$K_1(f_1 + f_2;t) \leq K_1(f_1;t) + K_1(f_2;t).$$

(v) For each $t \geq 0$ fixed, $K_1(\cdot;t)$ is a seminorm on $C[a,b]$, such that

$$K_1(f;t) \leq \|f\|_\infty$$

holds, for all $f \in C[a,b]$.

(vi) For a fixed $f \in C[a,b]$ and $t \geq 0$, the identity $K_1(|f|;t) = K_1(f;t)$ is true.

The following equivalence relation gives a first important link between the K-functional and the moduli (see [66]).

Theorem 1.1.5. *There exist constants c_1 and c_2 depending only on the integer $s = 1$ and $[a,b]$, such that*

$$c_1 \cdot \omega_1(f;t) \leq K_1(f;t) \leq c_2 \cdot \omega_1(f;t),$$

for all $f \in C[a,b]$ and $t > 0$.

Remark 1.1.6. In general, for $s \geq 1$, no sharp constants c_1 and c_2 are known, that satisfy the above inequality. In particular, for the cases $s = 1, 2$, so also for our case, such sharp constants exist, as we will see in the following result known as Brudnyĭ's representation theorem. This theorem is of crucial importance for the rest of this thesis, as a nice connection between $K_1(f;t)_{[a,b]}$ and the least concave majorant defined in (1.1.1).

Lemma 1.1.7. *Every function $f \in C[a,b]$ satisfies the equality*

$$K(f;t;C[a,b],C^1[a,b]) = \frac{1}{2} \cdot \widetilde{\omega}(f;2t),\ t \geq 0. \tag{1.1.2}$$

For details and proofs concerning this lemma, see R. Păltănea's article [91], the book [103], the book of R.T. Rockafellar [98] or the book [32].

The above equality (1.1.2) can be written in the following way

$$K\left(f;\frac{t}{2};C[a,b],C^1[a,b]\right) = \frac{1}{2} \cdot \widetilde{\omega}(f;t),\ t \geq 0.$$

1.1.3 Moduli of univariate functions defined on compact metric spaces

In this subsection we consider real-valued, continuous functions of one variable and recall definitions and properties in compact metric spaces. Let $f \in C(X) = C_{\mathbb{R}}(X,d)$, where $C_{\mathbb{R}}(X,d)$ is the space of all real-valued and continuous functions defined on the compact metric space (X,d), with diameter $d(X) > 0$.

We have the following definition for the (metric) modulus of continuity (see [45]) and its least concave majorant. This is a generalization of Definition 1.1.1 and equality (1.1.1).

Definition 1.1.8. Let $f \in C(X)$. If, for $t \in [0,\infty)$, the quantity

$$\omega_d(f;t) := \sup\left\{|f(x) - f(y)|\,; x,y \in X,\ d(x,y) \leq t\right\}$$

is the (metric) modulus of continuity, then its least concave majorant is given by

$$\widetilde{\omega_d}(f;t) = \begin{cases} \sup\limits_{0 \leq x \leq t \leq y \leq d(X), x \neq y} \frac{(t-x)\omega_d(f;y)+(y-t)\omega_d(f;x)}{y-x} & \text{for } 0 \leq t \leq d(X)\,, \\ \omega_d(f;d(X)) & \text{if } t > d(X)\,. \end{cases}$$

1.1.4 K-functionals and the connection to the moduli

For $0 < r \leq 1$, let Lip_r be the set of all functions $g \in C(X)$ with the property that

$$|g|_{Lip_r} := \sup_{d(x,y)>0} |g(x) - g(y)|\,/d^r(x,y) < \infty.$$

Lip_r is a dense subspace of $C(X)$ equipped with the supremum norm $\|\cdot\|_\infty$, and $|\cdot|_{Lip_r}$ is a seminorm on Lip_r. We also need to define the K-functional with respect to $(Lip_r, |\cdot|_{Lip_r})$, which is given by

$$K(t;f;C(X),Lip_r) := \inf_{g \in Lip_r}\{\|f-g\|_\infty + t \cdot |g|_{Lip_r}\},$$

for $f \in C(X)$ and $t \geq 0$.

The lemma of Brudnyĭ [83] that gives the relationship between the K-functional and the least concave majorant of the (metric) modulus of continuity will also be used in the proofs that follow.

Lemma 1.1.9. *Every continuous function f on X satisfies*

$$K\left(\frac{t}{2}; f; C(X), Lip_1\right) = \frac{1}{2} \cdot \widetilde{\omega_d}(f;t),\ 0 \leq t \leq d(X).$$

For more details about the (metric) moduli of smoothness, see [45].

1.1.5 Moduli of continuity of functions defined on the product of two metric spaces

In this section we consider products of two compact metric spaces and describe various moduli of smoothness (continuity) of functions defined on such products. For more details about parametric extensions and tensor products, see [45] and [30].

We take (X, d_X) and (Y, d_Y) two compact metric spaces. The cartesian product $X \times Y$, equipped with the product topology, and $d_{X\times Y}$ a metric on $X \times Y$ that generates this topology, is also a compact metric space. The metric satisfies the following properties:

$$d_{X\times Y}((x,r),(\hat{x},r)) = d_X(x,\hat{x}) \text{ for all } r \in Y \text{ and for all } (x,\hat{x}) \in X^2,$$
$$d_{X\times Y}((s,y),(s,\hat{y})) = d_Y(y,\hat{y}) \text{ for all } s \in X \text{ and for all } (y,\hat{y}) \in Y^2.$$

Also for the metric $d_{X\times Y}$ it holds

$$d_{X\times Y}((x,y),(\hat{x},\hat{y})) = d_X(x,\hat{x}) + d_Y(y,\hat{y}),$$

for $(x,y),(\hat{x},\hat{y}) \in X \times Y$.

These properties insure that we have interesting relationships between $\omega_{d_{X\times Y}}$ and moduli defined using d_X and d_Y, as we will see in the sequel.

We now define the total modulus of continuity of a function $f \in C(X \times Y)$.

Definition 1.1.10. For any function $f \in C(X \times Y)$ and $t_1, t_2 \in \mathbb{R}_+$, the total modulus of continuity of f with respect to d_X and d_Y is given by

$$\omega_{total,d_X,d_Y}(f;t_1,t_2) := \sup\{|f(x,y) - f(\hat{x},\hat{y})| : d_X(x,\hat{x}) \leq t_1, d_Y(y,\hat{y}) \leq t_2\}.$$

Regarding the relationship between $\omega_{X\times Y}$ and the total modulus of continuity, we have the following result.

Proposition 1.1.11. *(see Lemma 2.2 in [45]) Let (X, d_X), (Y, d_Y) and $d_{X\times Y}$ given as above. Then for any $t_1, t_2 \in \mathbb{R}_+$ we have*

$$\omega_{total,d_X,d_Y}(f;t_1,t_2) \leq \omega_{d_{X\times Y}}(f;t_1+t_2).$$

In our applications we will mostly use the Euclidean metric. For $x = (x_i)_{i=1}^m$ and $y = (y_i)_{i=1}^m$, this is given by

$$d_2(x,y) = \left(\sum_{i=1}^m (x_i - y_i)^2\right)^{\frac{1}{2}}.$$

The Euclidean metric can be generalized, in the sense that we get the more general metrics d_p, $1 \le p \le \infty$, given by

$$d_p(x,y) = \left(\sum_{i=1}^{m} |x_i - y_i|^p \right)^{\frac{1}{p}}, 1 \le p < \infty \text{ and}$$
$$d_\infty(x,y) = \max\{|x_i - y_i| : 1 \le i \le m\}.$$

In the same way, one can obtain different metrics, all generating the same topology and corresponding moduli, denoted by ω_{d_p}.

Remark 1.1.12. For details concerning the relationship between moduli of smoothness and K-functionals of different orders in the bivariate (multivariate) case, we give as references the book of L.L. Schumaker [102] and the references therein.

1.2 Positive linear operators

We give some definitions and properties regarding positive linear operators (see [110]). Some examples of such operators will also be considered, operators that will be used in order to illustrate our results.

Definition 1.2.1. Let X, Y be two linear spaces of real functions. Then the mapping $L : X \to Y$ is a linear operator if

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g),$$

for all $f, g \in X$ and $\alpha, \beta \in \mathbb{R}$. If for all $f \in X$, $f \ge 0$, it follows $Lf \ge 0$, then L is a positive operator.

X and Y can be different kinds of spaces, as we will show in the sequel.

Proposition 1.2.2 (Properties of positive linear operators)**.**
Let $L : X \to Y$ be a positive linear operator. Then we have the following inequalities:

i) If $f, g \in X$ with $f \le g$, then $Lf \le Lg$.

ii) For all $f \in X$, we have $|Lf| \le L|f|$.

Definition 1.2.3. Let $L : X \to Y$, where $X \subseteq Y$ are two linear normed spaces of real functions. To each operator L we assign a non-negative number $\|L\|$, given by

$$\|L\| := \sup_{f \in X, \|f\|=1} \|Lf\| = \sup_{f \in X, 0 < \|f\| \le 1} \frac{\|Lf\|}{\|f\|}.$$

We make a convention, that if X is the zero linear space, then any operator $L : X \to Y$ must be the zero operator and has the zero norm assigned to it.

$\|\cdot\|$ is called the operator norm.

If we take $X = Y = C[a,b]$, we can state the following:

Corollary 1.2.4. *For $L : C[a,b] \to C[a,b]$ being positive and linear, it follows that L is also continuous and it holds:*

$$\|L\| = \|Le_0\|.$$

1.2.1 The Bernstein operator

The Bernstein operator is maybe the most well-known example of a positive linear operator. It was introduced by S. N. Bernstein in 1912 (see [15]) and it was used to prove the fundamental theorem of Weierstrass (see [119]). For properties and further details about the Bernstein polynomials, see the book of R. A. deVore and G. G. Lorentz [32].

Considering the degree $n \in \mathbb{N}$, $x \in [0,1]$ and a function $f \in \mathbb{R}^{[0,1]}$, we have the following definition:

Definition 1.2.5. The $n-$th degree Bernstein polynomial $B_n f : [0,1] \to \mathbb{R}$ of the function f is defined by

$$B_n f := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot b_{n,k},$$

where the Bernstein fundamental functions are

$$b_{n,k}(x) := \begin{cases} \binom{n}{k} \cdot x^k (1-x)^{n-k} & \text{, for } 0 \leq k \leq n, \\ 0 & \text{, otherwise.} \end{cases}$$

Proposition 1.2.6. *(Properties of the Bernstein operator (see [32]))*

a) $\mathbb{R}^{[0,1]}$ *is endowed with the canonical operations of addition and scalar multiplication for functions, so the Bernstein operator* B_n *is a linear operator from* $\mathbb{R}^{[0,1]}$ *onto the subspace* $\Pi_n[0,1]$ *of polynomials of highest degree n on the interval* $[0,1]$.

b) The Bernstein operator is discretely defined, since $B_n f$ *only depends on the (n+1) function values* $f\left(\frac{k}{n}\right)$, $0 \leq k \leq n$.

c) For the case $n = 0$*, the Bernstein operator is not defined. Sometimes, it is set to be*

$$B_0 f := f(0).$$

d) If f *is non-negative, then this also holds for* $B_n f$.

e) We have endpoint interpolation as follows:

$$B_n(f;0) = f(0), B_n(f;1) = f(1).$$

f) The Bernstein operator reproduces linear polynomials, i.e., for every linear polynomial $L \in \Pi_1[0,1]$*, we have*

$$B_n L = L.$$

Remark 1.2.7. Because of the above proposition, we say that the Bernstein operator is a positive, linear operator.

The second moment of the Bernstein operator is given in the sequel.

Proposition 1.2.8. *It is well known that the second moment of the Bernstein polynomial is equal to*

$$B_n((e_1 - x)^2; x) = \frac{x(1-x)}{n},$$

where $e_i(x) = x^i$*, for* $i \geq 0$.

Definition 1.2.9 (Forward differences of $r-$th order with increment n)**.** Let (a_k) a finite or infinite sequence of real numbers. For suitable indices k and n, we denote with $\Delta_n a_k$ the difference $a_{k+n} - a_k$ between two elements of a sequence with difference (step size) n. More generally, for suitable $r \in \mathbb{N}_0$, denote

$$\Delta_n^r a_k := \begin{cases} a_k, & \text{if } r = 0, \\ \Delta_n(\Delta_n^{r-1} a_k), & \text{otherwise,} \end{cases}$$

the (Forward-) Difference of $r-$ Order with increment step (step size) n.

Proposition 1.2.10 (Derivatives of the Bernstein Polynomial)**.** *Let $0 \leq r \leq n$. Then the $r-$th derivative $(B_n f)^{(r)}$ of the $n-$th Bernstein polynomial has the form:*

$$(B_n f)^{(r)} = \sum_{k=0}^{n-r} n^{\underline{r}} \Delta_{\frac{1}{n}}^r f\left(\frac{k}{n}\right) b_{n-r,k},$$

where $n^{\underline{r}} = n(n-1) \cdot \ldots \cdot (n-r+1)$ is the $r-$th decreasing factorial of n terms.

Corollary 1.2.11. *We obtain for the Bernstein polynomial of the first derivative in the endpoints:*

a) $(B_n f)'(0) = \frac{f\left(\frac{1}{n}\right) - f(0)}{\frac{1}{n}}$

b) $(B_n f)'(1) = \frac{f(1) - f\left(1 - \frac{1}{n}\right)}{\frac{1}{n}}$

1.2.2 King operators

P. P. Korovkin [71] introduced in 1960 a result stating that if (L_n) is a sequence of positive linear operators on $C[a,b]$, then

$$\lim_{n\to\infty} L_n(f)(x) = f(x)$$

for each $f \in C[a,b]$ holds, if and only if

$$\lim_{n\to\infty} L_n(e_i(x)) = e_i(x)$$

for the three functions $e_i(x) = x^i$, $i = 0,1,2$. There are a lot of well-known operators, like the Bernstein, the Mirakjan-Favard-Szász and the Baskakov operators, that preserve e_0 and e_1 (see [67]). However, these operators do not reproduce e_2. We are now interested in a non-trivial sequence of positive linear operators (L_n) defined on $C[0,1]$, that preserve e_0 and e_2:

$$L_n(e_0)(x) = e_0(x) \text{ and } L_n(e_2)(x) = e_2(x),\ n = 0,1,2,\ldots.$$

In [67] J. P. King defined the following operators.

Definition 1.2.12. Let $(r_n(x))$ be a sequence of continuos functions with $0 \leq r_n(x) \leq 1$. Let $V_n : C[0,1] \to C[0,1]$ be given by:

$$\begin{aligned} V_n(f;x) &= \sum_{k=0}^{n} \binom{n}{k} (r_n(x))^k (1 - r_n(x))^{n-k} f\left(\frac{k}{n}\right) \\ &= \sum_{k=0}^{n} v_{n,k}(x) \cdot f\left(\frac{k}{n}\right), \end{aligned}$$

for $f \in C[0,1]$, $0 \leq x \leq 1$. $v_{n,k}$ are the fundamental functions of the V_n operator.

Remark 1.2.13. For $r_n(x) = x$, $n \in \mathbb{N}$, the positive linear operators V_n given above reduce to the Bernstein operator.

Proposition 1.2.14 (Properties of V_n).

1. *$V_n(e_0) = 1$ and $V_n(e_1; x) = r_n(x)$;*
2. *$V_n(e_2; x) = \frac{r_n(x)}{n} + \frac{n-1}{n}(r_n(x))^2$;*
3. *$\lim\limits_{n\to\infty} V_n(f; x) = f(x)$ for each $f \in C[0,1]$, $x \in [0,1]$, if and only if*
$$\lim_{n\to\infty} r_n(x) = x.$$
4. *The second moment in the general case is given by*
$$\begin{aligned} V_n((e_1 - x)^2; x) &:= \frac{r_n(x)}{n} + \frac{n-1}{n}(r_n(x))^2 - 2x r_n(x) + x^2 \\ &= \frac{1}{n} r_n(x)(1 - r_n(x)) + (r_n(x) - x)^2, \end{aligned} \tag{1.2.1}$$
where $0 \le r_n(x) \le 1$ are continuous functions.

For special ("right") choices of $r_n(x) = r_n^*(x)$, J. P. King showed in [67] that the following theorem holds.

Theorem 1.2.15. *(see Theorem 1.3. in [50]) Let $(V_n^*)_{n\in\mathbb{N}}$ be the sequence of operators defined before with*

$$r_n^*(x) := \begin{cases} r_1^*(x) = x^2, & \text{for } n = 1, \\ r_n^*(x) = -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}, & \text{for } n = 2, 3, \dots \end{cases}$$

Then we get $V_n^*(e_2; x) = x^2$, for $n \in \mathbb{N}$, $x \in [0,1]$ and $V_n^*(e_x; x) \neq e_1(x)$. V_n^* is not a polynomial operator.

The fundamental functions of this operator, namely

$$v_{n,k}^*(x) = \binom{n}{k}(r_n^*(x))^k(1 - r_n^*(x))^{n-k},$$

satisfy $\sum_{k=0}^{n} v_{n,k}^*(x) = 1$, for $n = 1, 2, \dots$.

Proposition 1.2.16 (Properties of r_n^*).

i) *$0 \le r_n^*(x) \le 1$, for $n = 1, 2, \dots$, and $0 \le x \le 1$.*

ii) *$\lim\limits_{n\to\infty} r_n^*(x) = x$ for $0 \le x \le 1$.*

The second moment of the special King-type operators V_n^* is given by

$$V_n^*((e_1 - x)^2; x) = 2x(x - r_n^*(x)),$$

so we discriminate between two cases.

The first case is $n = 1$, so $r_n^*(x) = x^2$ and the second moment is

$$V_1^*((e_1 - x)^2; x) = 2x^2(1 - x).$$

For the second case, $n = 2, 3, \ldots$, we have

$$r_n^*(x) = -\frac{1}{2(n-1)} + \sqrt{\left(\frac{n}{n-1}\right) x^2 + \frac{1}{4(n-1)^2}},$$

so the second moment is (see Theorem 2.2.14):

$$V_n^*((e_1 - x)^2; x) = 2x(x - r_n^*(x)) = 2x(x - V_n^*(e_1; x)).$$

The interest is now in finding r_n, such that the second moment is minimal. Such an approach was given in the thesis of P. Piţul [93]. There it was proven, that if the function

$$r_n^{min}(x) := \begin{cases} 0 & , x \in [0, \frac{1}{2n}) \\ \frac{2nx-1}{2n-2} & , x \in [\frac{1}{2n}, 1 - \frac{1}{2n}] \\ 1 & , x \in (1 - \frac{1}{2n}, 1] \end{cases}$$

is given, then the minimum value of the second moment is obtained. For the minimal second moments of V_n^{min}, the following representation was given

$$V_n^{min}((e_1 - x)^2; x) := \begin{cases} x^2 & , x \in [0, \frac{1}{2n}) \\ \frac{1}{n-1}(x(1-x) - \frac{1}{4n}) & , x \in [\frac{1}{2n}, 1 - \frac{1}{2n}] \\ (1-x)^2 & , x \in (1 - \frac{1}{2n}, 1] \end{cases}$$

1.2.3 The Bernstein-Stancu operator

The following generalization of the classical Bernstein operators was introduced by D.D. Stancu in 1972 (see [109], [49]). For α, β, γ positive numbers with $\alpha \geq 0$ and $0 \leq \beta \leq \gamma$, the definition of the Bernstein-Stancu positive linear operators $S_n^{<\alpha,\beta,\gamma>} : C[0,1] \to \Pi_n$ is:

$$S_n^{<\alpha,\beta,\gamma>}(f; x) := \sum_{k=0}^{n} s_{n,k}^{(\alpha)}(x) \cdot f\left(\frac{k+\beta}{n+\gamma}\right), \ x \in [0,1], \tag{1.2.2}$$

where $s_{n,k}^{(\alpha)}(x)$ are the fundamental polynomials

$$s_{n,k}^{(\alpha)}(x) := \binom{n}{k} \frac{x^{[k,-\alpha]} \cdot (1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}}, x \in [0,1], k = 0, \ldots, n.$$

Here $x^{[k,-\alpha]}$ is the factorial power of order k with step $-\alpha$ of x, i.e.,

$$\begin{aligned} x^{[0,-\alpha]} &= 1, \\ x^{[k,-\alpha]} &= x \cdot (x + \alpha) \cdot \ldots \cdot (x + (k-1)\alpha), \ k \in \mathbb{N}. \end{aligned}$$

When $\alpha = \beta = \gamma = 0$, we obtain the definition for the Bernstein operators. This is the reason why they are called "Bernstein-Stancu"-type operators.

We are interested in the case $\alpha = 0$. Then the operators $S_n^{<0,\beta,\gamma>}$ can be written according to (1.2.2) as

$$S_n^{<0,\beta,\gamma>}(f;x) = \sum_{k=0}^{n} b_{n,k}(x) \cdot f\left(\frac{k+\beta}{n+\gamma}\right),$$

for $b_{n,k}$ the fundamental Bernstein polynomials.

One important result that we will use later on involves powers of the operator $S_n^{<0,\beta,\gamma>}$. For a detailed proof see the proof of Theorem 4.32 in [93].

Theorem 1.2.17. *If $n \in \mathbb{N}$ is fixed, then for all $f \in C[0,1]$, $x \in [0,1]$*

$$\lim_{m\to\infty} \left[S_n^{<0,\beta,\gamma>}\right]^m (f;x) = b_0 e_0(x),$$

where $b_0 = b_0(f)$ is a convex combination of the values of the function f that appear in the operator's definition, namely

$$b_0 = \sum_{j=0}^{n} d_j f\left(\frac{j+\beta}{n+\gamma}\right),$$

with suitable $d_j \in \mathbb{R}$.

1.2.4 The Bernstein-Durrmeyer operator with Jacobi weights

The Durrmeyer operators, which were introduced by J.L. Durrmeyer in 1967 in his thesis [34], were given on $L^2[0,1]$ as a modification of the Bernstein operators. They were then generalized as follows:

We consider the Jacobi weight on $(0,1)$ to be

$$w^{(\alpha,\beta)}(x) = x^{\alpha}(1-x)^{\beta}, \alpha, \beta > -1,$$

and denote $L^1_{w^{(\alpha,\beta)}}(0,1)$ the space of Lebesgue-measurable functions f on $(0,1)$, such that the norm

$$\|f\|_{w^{(\alpha,\beta)}} := \sqrt{\int_0^1 f^2(x) w^{(\alpha,\beta)}(x)\mathrm{d}x}$$

is finite.

The operators $D_n^{(\alpha,\beta)} : L^1_{w^{(\alpha,\beta)}}(0,1) \to C[0,1]$ are defined by

$$D_n^{(\alpha,\beta)}(f;x) := \sum_{k=0}^{n} b_{n,k}(x) \cdot \frac{\int_0^1 b_{n,k}(t) f(t) w^{(\alpha,\beta)}(t)\mathrm{d}t}{\int_0^1 b_{n,k}(t) w^{(\alpha,\beta)}(t)\mathrm{d}t},$$

where $b_{n,k}$ are the Bernstein fundamental functions, and they are called the generalized Durrmeyer operators w.r.t. the Jacobi weight $w^{(\alpha,\beta)}$.

They are also called Bernstein-Jacobi operators because for any function $f \in C[0,1]$, $D_n^{(\alpha,\beta)} f$ can be written as a linear combination of Jacobi polynomials. These operators have the properties of being self-adjoint and commutative, properties inherited from the classical Durrmeyer operators. More details and properties can be found in [90], [13], [14].

In order to investigate the behaviour of the over-iterates of the Bernstein-Durrmeyer operator, we recall the following theorem that was proven in [93].

Theorem 1.2.18. *(see Theorem 4.40 in [93]) If $n \in \mathbb{N}$ is fixed, then for all f integrable on $[0,1]$ and $x \in [0,1]$, we have*

$$\lim_{m\to\infty}\left[D_n^{(\alpha,\beta)}\right]^m(f;x) = \left(\int_0^1 f(t)\cdot t^{\alpha}\cdot(1-t)^{\beta}\mathrm{dt}\right)e_0(x).$$

As one can conclude from the above result, the over-iterates of the operator tend toward a constant function.

1.2.5 The Hermite-Fejér interpolation operator

L. Fejér (see [38]) gave one proof of Weierstrass's approximation theorem by means of interpolation polynomials. We recall here his result as follows.

The classical Hermite-Fejér interpolation operator is a positive linear operator defined by

$$H_{2n-1}(f;x) := \sum_{k=1}^{n} f(x_k)(1-x\cdot x_k)\left(\frac{T_n(x)}{n(x-x_k)}\right)^2,$$

where $f \in C[-1,1]$ and $x_k = \cos\left(\frac{2k-1}{2n}\pi\right), 1 \le k \le n$, are the zeroes of $T_n(x) = \cos(n\cdot\arccos(x))$, $-1 \le x \le 1$, the $n-$th Chebyshev polynomial of the first kind.

Remark 1.2.19 (Properties of H_{2n-1}, see [38], [59]).

- $H_{2n-1}(f,x)$ is the unique polynomial of degree $\le 2n-1$ such that

  $$H_{2n-1}(f;x_k) = f(x_k),\ \text{for } k = 1,2,\ldots,n,$$

  and

  $$H'_{2n-1}(f;x_k) = 0,\ k = 1,2,\ldots,n.$$

- It is well-known that, for all $x \in [-1,1]$,

  $$(1-x\cdot x_k)\left(\frac{T_n(x)}{n(x-x_k)}\right)^2 \ge 0$$

  and

  $$\sum_{k=1}^{n}(1-x\cdot x_k)\left(\frac{T_n(x)}{n(x-x_k)}\right)^2 = 1.$$

The result of L. Fejér is presented next.

Theorem 1.2.20. *(L. Fejér, [38]) If $f \in C[-1,1]$ then $\lim_{n\to\infty}\|H_{2n-1}(f)-f\| = 0$, where $\|\cdot\|$ denotes the uniform norm on the space $C[-1,1]$.*

Next, let x_j be the node nearest to x, for $-1 \le x \le 1$. If two such nodes exist, let x_j be either of them.

Another result we will need in the sequel is a lemma given by O. Kiš (see [68], p. 30).

Lemma 1.2.21. *(O. Kiš) Let $-1 \le x = \cos\theta \le 1$, $x_k = \cos\theta_k$, $\theta_k = \frac{2k-1}{2n} \cdot \pi$, for $k = 1, 2, \dots, n$ and x_j be the node closest to x. Then*

$$\left|\theta - \theta_j\right| \le \frac{\pi}{2n} \left|\cos n\theta\right|.$$

The second moment of this operator is given as follows.

$$\begin{aligned} H_{2n-1}((e_1 - x)^2; x) &= \sum_{k=1}^{n} (x_k - x)^2 \cdot (1 - x \cdot x_k) \cdot \left(\frac{T_n(x)}{n(x - x_k)}\right)^2 \\ &= \frac{1}{n^2} T_n^2(x) \underbrace{\sum_{k=1}^{n} (1 - x \cdot x_k)}_{=n} \\ &= \frac{1}{n} T_n^2(x). \end{aligned}$$

1.2.6 The quasi-Hermite-Fejér operator

The quasi Hermite-Fejér operators were first considered by P. Szász in [114]. The interest was to find a uniquely defined polynomial of degree less than or equal to $2n+1$, that satisfies the conditions

$$\begin{aligned} L(f; x_v) &= f(x_v), \text{ for } 0 \le v \le n+1, \\ L'(f; x_v) &= 0, \text{ for } 1 \le v \le n, \end{aligned}$$

for the fundamental nodes $x_1, \dots, x_n \in (-1, 1)$, $x_0 = -1$, $x_{n+1} = 1$.

The quasi-Hermite-Fejér interpolation operator $Q_n : C[-1, 1] \to \Pi_{2n+1}$ with arbitrary nodes has the form:

$$\begin{aligned} Q_n(f; x) := f(-1) \cdot \frac{1 - x}{2w(-1)^2} \cdot w(x)^2 + f(1) \cdot \frac{1 + x}{2w(1)^2} \cdot w(x)^2 + \\ + \sum_{v=1}^{n} f(x_v) \cdot \frac{1 - x^2}{1 - x_v^2} \cdot [1 + c_v(x - x_v)] \cdot \left(\frac{w(x)}{w'(x_v)(x - x_v)}\right)^2, \end{aligned}$$

for $w(x) = c \cdot \prod_{v=1}^{n} (x - x_v)$, $c \ne 0$ and

$$c_v = \frac{2x_v}{1 - x_v^2} - \frac{w''(x_v)}{w'(x_v)}, \ 1 \le v \le n.$$

For all $x \in [-1, 1]$, $1 + c_v(x - x_v) \ge 0$. From this inequality we can say that Q_n is a positive linear operator.

All of this holds especially for the zeroes of a Jacobi-Polynomial $P_n^{\alpha,\beta}$, for $0 \le \alpha, \beta \le 1$.

We are interested in the approximation behaviour of a special case of knots, meaning Chebyshev knots of the second kind. For the zeroes of the Jacobi-Polynomials $P_n^{\frac{1}{2},\frac{1}{2}}$ (for $\alpha = \beta = \frac{1}{2}$), the quasi-Hermite-Fejér interpolation polynomial has the form

$$Q_n(f;x) := \sum_{v=0}^{n+1} f(x_v) \cdot F_{n,v}(x) \cdot U_n^2(x),$$

where $U_n(x)$ is the $n-$th Chebyshev polynomial of the second kind with roots $x_v = \cos\left(\frac{v}{n+1} \cdot \pi\right)$, for $1 \le v \le n$ and with

$$F_{n,v}(x) := \begin{cases} \frac{1-x}{2(n+1)^2} & \text{, for } v = 0, \\ \frac{(1-x^2)(1-x_v \cdot x)}{(n+1)^2(x-x_v)^2} & \text{, for } 1 \le v \le n, \\ \frac{1+x}{2(n+1)^2} & \text{, for } v = n+1. \end{cases}$$

Q_n is a positive linear operator, for all $n \ge 1$.

It holds

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)},$$

and we have the relations:

$$U_n(\pm 1)^2 = (n+1)^2 \text{ and}$$
$$(1 - x_v^2) \cdot U_n'(x_v) = (-1)^{v+1}(n+1).$$

This operator also reproduces constant functions. The second moment of the operator is given by

$$Q_n((e_1 - x)^2; x) = (1 - x^2) \cdot \frac{U_n^2(x)}{n+1},$$

while the first moment is

$$Q_n(e_1 - x; x) = \frac{(1 - x^2)U_n(x)}{n+1}\{T_{n+1}(x) - x \cdot U_n(x)\},$$

for $T_{n+1}(x) = \cos((n+1)\arccos x)$.

1.2.7 The almost-Hermite-Fejér operator

The so-called almost-Hermite-Fejér interpolation was studied by many authors. For reference, we recall a paper in which a survey presenting results in this setting, including our particular case, is given, paper written by H. Gonska in 1982 [43].

Let us consider an $(r,s)-$Hermite-Fejér interpolation operator

$$F_{r,s;n} : C[-1,1] \to \Pi_{2n+r+s-1}$$

and the image of a function $f \in C[-1,1]$ under such an operator. Then we get the uniquely determined algebraic polynomial that, for a fixed sequence of nodes

$$1 = x_0 > x_1 > \ldots > x_n > x_{n+1} = -1,$$

satisfies the $2n + r + s$ conditions

$$F_{r,s;n}(f; x_v) = f(x_v), \quad (F_{r,s;n}f)'(x_v) = 0, \text{ for } 1 \le v \le n,$$
$$F_{r,s;n}(f; 1) = f(1) \text{ for } r \ge 1, \ (F_{r,s;n}f)^{(\rho)}(1) = 0, \text{ for } 1 \le \rho \le r-1,$$
$$F_{r,s;n}(f; -1) = f(-1) \text{ for } s \ge 1, \ (F_{r,s;n}f)^{(\sigma)}(-1) = 0, \text{ for } 1 \le \sigma \le s-1.$$

Given the fact that the nodes $x_1, x_2, \ldots, x_n \in (-1, 1)$ are distributed like this, it is natural to analyse $(r, s)-$Hermite-Fejér formulas based on the endpoints ± 1 and the roots $x_1, \ldots, x_n$ of the Jacobi polynomials $P_n^{\alpha,\beta}$, $\alpha, \beta > -1$. This process was treated by many authors (see [43], [117], [69], [81] and the references in these papers). Then the corresponding operators are $F_{r,s;n}^{(\alpha,\beta)}$. We consider the particular case $(r, s) = (1, 0)$ and the corresponding operators $F_{1,0;n}^{(\alpha,\beta)}$ are the almost-Hermite-Fejér-interpolation operators. They are given by the formula

$$F_{1,0;n}^{(\alpha,\beta)}(f; x) := f(1) \cdot \frac{w(x)^2}{w(1)^2} + \sum_{v=1}^{n} f(x_v) \cdot \frac{1-x}{1-x_v}[1 + c_v^*(x - x_v)] \cdot l_v(x)^2,$$

where l_v denotes the $v-$th Lagrange fundamental polynomial and

$$w(x) = \prod_{v=1}^{n} (x - x_v).$$

Furthermore,

$$c_v^* = \frac{1}{1 - x_v} - \frac{w''(x_v)}{w'(x_v)}.$$

If $1 + c_v^*(x - x_v) \ge 0$ for all $x \in [-1, 1]$, then the above operator is positive and linear. If the nodes $x_1, \ldots, x_n$ are the zeroes of a Jacobi polynomial $P_n^{(\alpha,\beta)}$, then this is the case for all n if and only if $(\alpha, \beta) \in [0, 1] \times (-1, 0]$.

One interesting case that we now cover is for $(\alpha, \beta) = \left(\frac{1}{2}, -\frac{1}{2}\right)$, in which case the corresponding operators are positive and the property of uniform convergence for every $f \in C[-1, 1]$ holds. For this choice of (α, β) the operators have the following form (see [115]):

$$F_{1,0;n}^{\left(\frac{1}{2},-\frac{1}{2}\right)}(f; x) := f(1) \cdot \frac{w(x)^2}{w(1)^2} + \sum_{v=1}^{n} f(x_v) \cdot \frac{1-x}{1-x_v} \cdot \frac{1 - x x_v}{1 - x_v^2} \cdot l_v(x)^2,$$

where

$$w(x) = \frac{\sin \frac{2n+1}{2} \arccos x}{\sin \frac{1}{2} \arccos x}, \ x_v = \cos \frac{2v}{2n+1}\pi, \ 1 \le v \le n,$$

and l_v is the vth Lagrange fundamental polynomial. For the above positive operators it holds

$$F_{1,0;n}^{\left(\frac{1}{2},-\frac{1}{2}\right)}(e_0; x) = 1.$$

For these operators $F_{1,0;n}^{\left(\frac{1}{2},-\frac{1}{2}\right)}$ based upon the roots of the Jacobi polynomials $P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}$ and the endpoint 1, we have for all $n \ge 2$ that the first absolute moment (see [43] and [73]) is given by

$$F_{1,0;n}^{\left(\frac{1}{2},-\frac{1}{2}\right)}(|e_1 - x|; x) \le c \cdot \frac{1 + \sqrt{1-x^2} \cdot \ln n}{2n+1},$$

for a suitable constant c. The second moment is given by the following equality (see [42]):

$$F_{1,0;n}^{\left(\frac{1}{2},-\frac{1}{2}\right)}((e_1-x)^2;x)=\frac{2(1-x)\cdot w(x)^2}{3n},$$

while the absolute value of the first moment (see [42]) satisfies

$$\begin{aligned}\left|F_{1,0;n}^{\left(\frac{1}{2},-\frac{1}{2}\right)}(e_1-x;x)\right| &= \frac{1}{(2n+1)^2}\cdot\left|(1-x)\cdot w(x)\cdot[2(1-x^2)\cdot w'(x)+(2nx-1)w(x)]\right| \\ &\leq \frac{4\sqrt{1-x}\cdot|w(x)|}{n},\ n\geq 2.\end{aligned}$$

1.2.8 Convolution-type operators

These types of operators were treated by many authors, like J. -D. Cao, H. Gonska and H. -J. Wenz (see [21]). The following concepts, as well as the given applications, can also be found in [101].

One of the first authors to give the following definition was H. G. Lehnhoff in [74]:

Definition 1.2.22. For the case $X=[-1,1]$, given a function $f\in C(X)$ and any natural number n, the convolution operator $G_{m(n)}$ is given by

$$G_{m(n)}(f;x):=\frac{1}{\pi}\cdot\int_{-\pi}^{\pi}f(\cos(\arccos(x)+v))\cdot K_{m(n)}(v)\mathrm{d}v,$$

where the kernel $K_{m(n)}$ is a positive and even trigonometric polynomial of degree $m(n)$ satisfying

$$\int_{-\pi}^{\pi}K_{m(n)}(v)\mathrm{d}v=\pi,$$

meaning that $G_{m(n)}(e_0;x)=1$, for $x\in X$.

It is clear that $G_{m(n)}(f;\cdot)$ is an algebraic polynomial of degree $m(n)$ and the kernel $K_{m(n)}$ has the following form:

$$K_{m(n)}(v)=\frac{1}{2}+\sum_{k=1}^{m(n)}\rho_{k;m(n)}\cdot\cos(kv),$$

for $v\in[-\pi,\pi]$.

We also need another result that goes back to H.G. Lehnhoff [74]:

Lemma 1.2.23. *For $x\in X$, the equality*

$$G_{m(n)}((e_1-x)^2;x)=x^2\left\{\frac{3}{2}-2\cdot\rho_{1;m(n)}+\frac{1}{2}\cdot\rho_{2;m(n)}\right\}+(1-x^2)\cdot\left\{\frac{1}{2}-\frac{1}{2}\cdot\rho_{2;m(n)}\right\}$$

holds. Here e_1 denotes the first monomial given by $e_1(t)=t$ for $|t|\leq 1$.

The first moment of the convolution-type operator (see [20]) is given by:

$$G_{m(n)}(e_1-x;x)=x\cdot[\rho_{1;m(n)}-1].$$

The above lemma gives the second moment of the convolution-type operator, which, along with the first moment, will be needed in the sequel.

Furthermore, we take into account different degrees $m(n)$ and different convolution operators, respectively.

1.2.8.1 Convolution operators with Fejér-Korovkin kernel

If we consider degree $m(n) = n-1$, for $n \in \mathbb{N}$, the Fejér-Korovkin kernel is given by

$$K_{n-1}(v) = \frac{1}{n+1}\left(\frac{\sin\left(\frac{\pi}{n+1}\right)\cdot\cos\left((n+1)\frac{v}{2}\right)}{\cos(v)-\cos\left(\frac{\pi}{n+1}\right)}\right)^2$$

with

$$\rho_{1;n-1} = \cos\left(\frac{\pi}{n+1}\right),\ \rho_{2;n-1} = \frac{n}{n+1}\cos\left(\frac{2\pi}{n+1}\right)+\frac{1}{n+1}.$$

Using the latter relations, we get

$$\begin{aligned}
G_{n-1}\left((e_1-x)^2;x\right) &\le \left|\frac{3}{2}-2\cdot\rho_{1;n-1}+\frac{1}{2}\rho_{2;n-1}\right|+\frac{1}{2}\left|1-\rho_{2;n-1}\right| \\
&\le \left|\frac{3}{2}-2\cdot\cos\left(\frac{\pi}{n+1}\right)+\frac{1}{2(n+1)}+\frac{n}{2(n+1)}\cdot\cos\left(\frac{2\pi}{n+1}\right)\right| \\
&+\frac{1}{2}\cdot\left|1-\frac{1}{n+1}-\frac{n}{n+1}\cdot\cos\left(\frac{2\pi}{n+1}\right)\right| \\
&\le 3\cdot\left(\frac{\pi}{n+1}\right)^2+\left(\frac{\pi}{n+1}\right)^2 \\
&= 4\cdot\left(\frac{\pi}{n+1}\right)^2.
\end{aligned}$$

1.2.8.2 Convolution operators with de La Vallée Poussin kernel

We now have degree $m(n) = n \in \mathbb{N}_0$ and we define the de La Vallée Poussin kernel by

$$V_n(v) = \frac{(n!)^2}{(2n)!}\cdot\left(2\cos\left(\frac{v}{2}\right)\right)^{2n},$$

with

$$\rho_{1;n} = \frac{n}{n+1},\ \rho_{2;n} = \frac{(n-1)n}{(n+1)(n+2)}.$$

Using the two relations, we have for the second moment:

$$\begin{aligned}
G_n\left((e_1-x)^2;x\right) &\le \left|\frac{3}{2}-\frac{2n}{n+1}+\frac{1}{2}\cdot\frac{n(n-1)}{(n+1)(n+2)}\right| \\
&+\frac{1}{2}\left|1-\frac{n(n-1)}{(n+1)(n+2)}\right| \\
&\le \left|\frac{3}{(n+1)(n+2)}\right|+\left|\frac{2n+1}{(n+1)(n+2)}\right| \\
&\le \frac{2}{n+1}.
\end{aligned}$$

We also know

$$G_n(e_1;x) = \rho_{1;n}\cdot x = \frac{n}{n+1}\cdot x,$$

which implies that

$$G_n(e_1;x) - x = \frac{n}{n+1} \cdot x - x = x \cdot \left(\frac{n}{n+1} - 1\right) = -x \cdot \frac{1}{n+1}.$$

1.2.8.3 Convolution operators with Jackson kernel

Finally, the last operator we consider is of degree $m(n) = 2n - 2$, with $n \in \mathbb{N}$. For this, the Jackson kernel has the form

$$J_{2n-2}(v) = \frac{3}{2n(2n^2+1)} \cdot \left(\frac{\sin\left(n\frac{v}{2}\right)}{\sin\left(\frac{v}{2}\right)}\right)^4,$$

with

$$\rho_{1;2n-2} = \frac{2n^2-2}{2n^2+1},\ \rho_{2;2n-2} = \frac{2n^3-11n+9}{n(2n^2+1)},$$

and the second moment satisfying

$$\begin{aligned} G_{2n-2}\left((e_1-x)^2;x\right) &\leq \left|\frac{3}{2} - \frac{4n^2-4}{2n^2+1} + \frac{1}{2} \cdot \frac{2n^3-11n+9}{n(2n^2+1)}\right| \\ &\quad + \frac{1}{2} \cdot \left|1 - \frac{2n^3-11n+9}{n(2n^2+1)}\right| \\ &\leq \left|\frac{9}{2n(2n^2+1)}\right| + \left|\frac{12n-9}{2n(2n^2+1)}\right| \\ &\leq \frac{6}{2n^2+1} \leq \frac{3}{n^2}. \end{aligned}$$

1.2.9 Shepard-type operators

We present some Shepard-type operators defined in the general setting. An example of such operators goes back to the work of I. K. Crain, B. K. Bhattacharyya [28] and D. Shepard [104] and was first investigated by W. J. Gordon and J. A. Wixom [60]. Other important references are, e.g., the Habilitationsschrift [45] and the paper [44], both of H. Gonska. In both of the latter references, we have the following:

Definition 1.2.24. (see Definition 3.2. in [100]) Let (X,d) be a metric space and let $x_1,\ldots,x_n$ be a finite collection of distinct points in X. We further suppose that for each n-tuple $(x_1,\ldots,x_n)$ we have a finite given sequence $(\mu_1,\ldots,\mu_n)$ of real numbers $\mu_i > 0$. Then the Crain-Bhattacharyya-Shepard (CBS) operator is given by

$$S_n(f;x) := S_{x_1,\ldots,x_n}^{\mu_1,\ldots,\mu_n}(f;x) := \begin{cases} \sum\limits_{i=1}^{n} f(x_i) \cdot \frac{d(x,x_i)^{-\mu_i}}{\sum\limits_{l=1}^{n} d(x,x_l)^{-\mu_l}} & , x \notin \{x_1,\ldots,x_n\}, \\ f(x_i) & , \text{otherwise.} \end{cases}$$

Here $x \in X$ and f is a real-valued function defined on X.

Remark 1.2.25. (see Remark 3.3. in [100])

From the above definition, we can state that S_n is a positive linear operator on $C(X)$ that satisfies $S_n(1_X; x) = 1$, for all $x \in X$. Also it holds that $S_n(f; x_i) = f(x_i)$, for all $x_i, 1 \leq i \leq n$.

If we restrict ourselves to the simpler case $1 \leq \mu = \mu_1 = \ldots = \mu_n$, we denote the corresponding operator by S_n^μ. This looks like

$$S_n^\mu(f;x) := \begin{cases} \sum\limits_{i=1}^{n} f(x_i) \cdot \frac{d(x,x_i)^{-\mu}}{\sum\limits_{l=1}^{n} d(x,x_l)^{-\mu}} & , x \notin \{x_1, \ldots, x_n\}, \\ f(x_i) & , \text{otherwise}, \end{cases}$$

$$= \begin{cases} \sum\limits_{i=1}^{n} f(x_i) \cdot s_i^\mu(x) & , x \notin \{x_1, \ldots, x_n\}, \\ f(x_i) & , \text{otherwise}, \end{cases} \tag{1.2.3}$$

while the second moment of this CBS operator can be written as

$$S_n^\mu(d^2(\cdot, x); x) = \begin{cases} \sum\limits_{i=1}^{n} \frac{d(x,x_i)^{2-\mu}}{\sum\limits_{l=1}^{n} d(x,x_l)^{-\mu}} & , x \notin \{x_1, \ldots, x_n\}, \\ 0 & , \text{otherwise}. \end{cases} \tag{1.2.4}$$

For a second special case, we take $X = [0,1]$ and the metric $d(x,y) := |x - y|$, for $x, y \in X$. Then we get the CBS operator S_{n+1}^μ based on $n+1$ equidistant points $x_i = \frac{i}{n}, 0 \leq i \leq n$, given by

$$S_{n+1}^\mu(f;x) := \begin{cases} \sum\limits_{i=0}^{n} f(x_i) \cdot \frac{\left|x - \frac{i}{n}\right|^{-\mu}}{\sum\limits_{l=0}^{n} \left|x - \frac{l}{n}\right|^{-\mu}} & , x \notin \{x_0, \ldots, x_n\} \\ f(x_i) & , \text{otherwise}. \end{cases} \tag{1.2.5}$$

1.2.10 A piecewise linear interpolation operator S_{Δ_n}

We consider the operator $S_{\Delta_n} : C[0,1] \to C[0,1]$ (see [46]) interpolating the function at the points $0, \frac{1}{n}, \ldots, \frac{k}{n}, \ldots, \frac{n-1}{n}, 1$, which can be explicitely described as

$$S_{\Delta_n}(f;x) = \frac{1}{n} \sum_{k=0}^{n} \left[\frac{k-1}{n}, \frac{k}{n}, \frac{k+1}{n}; |\alpha - x| \right]_\alpha f\left(\frac{k}{n}\right),$$

where $[a,b,c;f] = [a,b,c;f(\alpha)]_\alpha$ denotes the divided difference of a function $f : D \to \mathbb{R}$ on the (distinct knots) $\{a,b,c\} \subset D$, $D \subset \mathbb{R}$, w.r.t. α.

Proposition 1.2.26 (Properties of S_{Δ_n})**.**

i) *S_{Δ_n} is a positive, linear operator preserving linear functions.*

ii) *S_{Δ_n} preserves monotonicity and convexity/concavity.*

iii) *$S_{\Delta_n}(f;0) = 0$, $S_{\Delta_n}(f;1) = f(1)$.*

iv) *If $f \in C[0,1]$ is convex, then $S_{\Delta_n} f$ is also convex and we have: $f \leq S_{\Delta_n} f$.*

The operator S_{Δ_n} can also be given as follows:

$$S_{\Delta_n} f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) u_{n,k}(x),$$

for $f \in C[0,1]$ and $x \in [0,1]$, where $u_{n,k} \in C[0,1]$ are piecewise linear and continuous functions, such that

$$u_{n,k}\left(\frac{l}{n}\right) = \delta_{kl},\ k,l = 0,\ldots,n.$$

We now give the second moment of the operator. For $x \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$, we have

$$\begin{aligned} S_{\Delta_n}((e_1 - x)^2; x) &= n\left(x - \frac{k-1}{n}\right)\left(\frac{k}{n} - x\right)\left[\left(\frac{k}{n} - x\right) - \left(\frac{k-1}{n} - x\right)\right] \\ &= \left(x - \frac{k-1}{n}\right)\left(\frac{k}{n} - x\right), \end{aligned}$$

which is maximal when $x = \frac{2k-1}{2n}$. This implies

$$S_{\Delta_n}((e_1 - x)^2; x) \leq \frac{1}{4n^2}.$$

1.2.11 The BLaC operator: Definitions and Properties

The idea to examine BLaC operators comes from the BLaC-wavelets (*Blending of Linear and Constant wavelets*), introduced around 1996 by G. P. Bonneau, S. Hahmann and G. Nielson (see [16]). They present a multiresolution analysis that implies a function representation at multiple levels of detail. This is a tool for handling large sets of data. The wavelet coefficients are the ones who store the loss of detail in each level of representation. The wavelets are basis functions encoding the difference between two succesive levels. Throughout their work, they discriminate among *Haar and linear wavelets*. The Haar wavelets are not continuous, but have perfect locality, while the linear ones are continuous, but the regularity they possess can be a drawback. A compromise between the locality of the analysis and the regularity of the approximation is desired.

This compromise is obtained by using a blending parameter $0 < \Delta \leq 1$. We now introduce the operator. The results that appear in the sequel are also present in [101].

Definition 1.2.27. (see [47]) For $f \in C[0,1]$ and $x \in [0,1]$, the *BLaC operator* is given by

$$BL_n(f; x) := \sum_{k=-1}^{2^n-1} f(\eta_k^n) \cdot \varphi_k^n(x).$$

Now we explain the definition.

For the real blending parameter, the *scaling functions* $\varphi_\Delta : \mathbb{R} \to [0,1]$ are given by

$$\varphi_\Delta := \begin{cases} \frac{x}{\Delta}, & \text{for } 0 \leq x < \Delta, \\ 1, & \text{for } \Delta \leq x < 1, \\ -\frac{1}{\Delta} \cdot (x - 1 - \Delta), & \text{for } 1 \leq x < 1 + \Delta, \\ 0, & \text{else}. \end{cases}$$

Remark 1.2.28. For $\Delta = 1$, φ_Δ reduces to B-Spline functions of first order (or *hat-functions*), while for the case $\Delta \to 0$ the piecewise constant functions are obtained. That's why we choose $\Delta \in (0, 1]$.

For the index $k = -1, \ldots, 2^n - 1$, $n \in \mathbb{N}$, by dilatation and translation of φ_Δ we obtain the *family of fundamental functions*:

$$\varphi_k^n(x) := \varphi_\Delta(2^n x - k),\ x \in [0, 1].$$

The *midpoints* η_k^n of the support line of each fundamental function φ_k^n are given by

$$\eta_k^n := \frac{k}{2^n} + \frac{1}{2} \cdot \frac{1+\Delta}{2^n},\ \text{for } k = 0, \ldots, 2^n - 2.$$

For $k \in \{-1, 2^n - 1\}$ let $\eta_{-1}^n := 0$ and $\eta_{2^n-1}^n := 1$.

Proposition 1.2.29. *(Properties of the BLaC operator, see [47])*

i) $BL_n : C[0, 1] \to C[0, 1]$ *is positive and linear;*

ii) BL_n *is a modification of the piecewise linear interpolation operator* $S_{\Delta_n,1}$*;*

iii) BL_n *interpolates* f *at* η_k^n, $k = -1, \ldots, 2^n - 1$ *(also at the endpoints* 0 *and* 1*);*

iv) BL_n *reproduces constant functions;*

v) The first absolute moment of the BLaC operator is:

$$BL_n(|e_1 - x|\,; x) = \sum_{k=-1}^{2^n-1} |\eta_k^n - x| \cdot \varphi_k^n(x) \le \frac{1}{2^n},\ \textit{for all } x \in [0, 1];$$

vi) The second moment of the BLaC operator is given by:

$$BL_n((e_1 - x)^2; x) = \sum_{k=-1}^{2^n-1} (\eta_k^n - x)^2 \cdot \varphi_k^n(x) \le \frac{1}{2^{2n}},\ \textit{for all } x \in [0, 1].$$

1.2.12 The Mirakjan-Favard-Szász operator

The Mirakjan-Favard-Szász operators (see e.g. [6]) were independently introduced by G.M. Mirakjan (see [82]) in 1941 and by J. Favard and O. Szász (see [37] and [113]). The classical $n-$th Mirakjan-Favard-Szász operator M_n is defined by

$$M_n(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1.2.6}$$

for $f \in E_2$, $x \in [0, \infty) \subset \mathbb{R}$ and $n \in \mathbb{N}$. E_2 is the Banach lattice

$$E_2 := \left\{ f \in C([0, \infty)) : \frac{f(x)}{1 + x^2} \text{ is convergent as } x \to \infty \right\},$$

endowed with the norm $\|f\|_* := \sup\limits_{x \ge 0} \frac{|f(x)|}{1+x^2}$.

The series on the right-hand side of (1.2.6) is absolutely convergent and E_2 is isomorphic to $C[0, 1]$; (see [6], Sect. 5.3.9).

1.2.13 The Baskakov operator

In the book of F. Altomare and M. Campiti [6] (Sect. 5.3.10), the classical positive, linear Baskakov operators $(A_n)_{n\in\mathbb{N}}$ are defined as follows:

$$A_n(f;x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}},$$

for every $f \in E_2$ and $x \in [0,\infty[$.

1.2.14 The Bleimann-Butzer-Hahn operator

In the same book [6] (Sect. 5.2.8), the Bleimann-Butzer-Hahn operators are also presented. For every $n \in \mathbb{N}$ the positive linear operator $BH_n : C_b([0,\infty)) \to C_b([0,\infty))$ is defined by

$$BH_n(f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\cdot bh_{n,k}(x),$$

for every $f \in C_b([0,\infty))$, $x \in [0,\infty)$, $n \in \mathbb{N}$.

The fundamental functions are given by

$$bh_{n,k}(x) := \frac{1}{(1+x)^n}\cdot\binom{n}{k}x^k.$$

1.3 Lagrange interpolation

Consider $f \in C[-1,1]$ and the infinite matrix $X = \{x_{k,n}\}_{k=1}^{n}{}_{n=1}^{\infty}$ with

$$-1 \le x_{1,n} < x_{2,n} < \ldots < x_{n,n} \le 1, \text{ for } n = 1,2,\ldots.$$

The Lagrange fundamental functions are given by

$$l_{k,n}(x) = \frac{w_n(x)}{w_n'(x_{k,n})(x-x_{k,n})},\ 1 \le k \le n,$$

where $w_n(x) = \prod_{k=1}^{n}(x-x_{k,n})$ and the Lagrange operator $L_n : C[-1,1] \to \Pi_{n-1}$ (see [112]) is

$$L_n(f;x) := \sum_{k=1}^{n} f(x_{k,n})l_{k,n}(x).$$

The Lebesgue function of the interpolation is:

$$\Lambda_n(x) := \sum_{k=1}^{n}|l_{k,n}(x)|,$$

and the Lebesgue constant is given by

$$\Lambda_n(X) := \max_{-1\le x\le 1}\Lambda_n(x).$$

It is also known (see [27], p. 13) that $\|L_n\| < \infty$ and

$$\|L_n\| = \|\Lambda_n\|_\infty$$

hold.

Proposition 1.3.1 (Properties of the Lagrange operator)**.**

i) The Lagrange operator is linear but only in exceptional cases positive.

ii) $L_n(f; x_{k,n}) = f(x_{k,n}),\ 1 \leq k \leq n.$

iii) The Lagrange operator is idempotent: $L_n^2 = L_n$.

iv) $\sum_{k=1}^{n} l_{k,n}(x) = 1$ *holds, so* L_n *satisfies* $L_n(e_0; x) = 1$.

Remark 1.3.2. The Lebesgue function has been studied for different node systems. In the sequel, we will use some known results for Chebyshev nodes.

First, we recall a result from [64] (see relation (3.1) there), for the fundamental functions of a Lagrange interpolation based upon any infinite matrix X. It holds, for $\alpha = 2$, that

$$\sum_{k=1}^{n} |l_{k,n}(x)|^2 \geq \frac{1}{4}, -1 \leq x \leq 1. \tag{1.3.1}$$

This was proven for a general $\alpha > 0$ using Lemma 4 in a paper by P.Erdös and P. Turán (see [36]).

We only consider the Lagrange operator with Chebyshev nodes (see [17], [27]) and the corresponding Lebesgue function.

Let $T_n(x) = \cos(n(\arccos x))$ and $X = \{\cos[\pi(2k-1)/2n]\}$, i.e., when

$$x_{k,n} = \cos\theta_k = \cos\frac{2k-1}{2n} \cdot \pi \ (k = 1, 2, \ldots, n; n =\in \mathbb{N})$$

are the Chebyshev roots. For each n these nodes are the zeroes of the n-th Chebyshev polynomial of the first kind.

In [107], the author illustrated the maximum of the Lebesgue function, which is attained in ± 1 (also see the citations in this paper). Asymptotic results can be given for the Lebesgue constant

$$\Lambda_n(X) = \frac{2}{\pi}\log n + \frac{2}{\pi} \cdot \left(\gamma + \log\frac{8}{\pi}\right) + \mathcal{O}\left(\frac{1}{n^2}\right),$$

where γ denotes Euler's constant $0,577....$ For more results and references, see [17] and [118].

2 Univariate Chebyshev-Grüss Inequalities

2.1 Auxiliary and historical results

Here we list some classical results which we will need in the sequel.

The functional given by

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \cdot \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x,$$

where $f,g : [a,b] \to \mathbb{R}$ are integrable functions, is well known in the literature as the classical Chebyshev functional (see [25]).

We first recall the following result.

Theorem 2.1.1. *(see [85]) Let $f,g : [a,b] \to \mathbb{R}$ be bounded integrable functions, both increasing or both decreasing. Furthermore, let $p : [a,b] \to \mathbb{R}_0^+$ be a bounded and integrable function. Then*

$$\int_a^b p(x)\mathrm{d}x \int_a^b p(x)\cdot f(x)\cdot g(x)\mathrm{d}x \geq \int_a^b p(x)\cdot f(x)\mathrm{d}x \int_a^b p(x)\cdot g(x)\mathrm{d}x. \tag{2.1.1}$$

If one of the functions f or g is nonincreasing and the other nondecreasing, then inequality (2.1.1) is reversed.

Remark 2.1.2. Inequality (2.1.1) is known as Chebyshev's inequality. It was first introduced by P. L. Chebyshev in 1882 in [24]. If $p(x) = 1$ for $a \leq x \leq b$, then inequality (2.1.1) is equivalent to

$$\frac{1}{b-a}\int_a^b f(x)\cdot g(x)\mathrm{d}x \geq \left(\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right)\cdot\left(\frac{1}{b-a}\int_a^b g(x)\mathrm{d}x\right).$$

We now cite another classical result involving the Chebyshev functional.

Theorem 2.1.3 (Chebyshev, 1882, see [9]). *Let $f,g \in [a,b] \to \mathbb{R}$ be absolutely continuous functions. If $f', g' \in L_\infty([a,b])$, then*

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \cdot \|f'\|_\infty \cdot \|g'\|_\infty .$$

The next result is the Grüss inequality for the Chebyshev functional.

Theorem 2.1.4. *(Grüss, 1935, see [62]) Let f,g be integrable functions from $[a,b]$ into $\mathbb{R}$, such that $m \leq f(x) \leq M$, $p \leq g(x) \leq P$, for all $x \in [a,b]$, where $m, M, p, P \in \mathbb{R}$. Then*

$$|T(f,g)| \leq \frac{1}{4}(M-m)(P-p).$$

The functional L, given by $L(f) := \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$, is linear and positive and satisfies $L(e_0) = 1$; here we denote $e_i(x) = x^i$, for $i \geq 0$. In the sequel, we recall some bounds for what we call the generalized Chebyshev functional

$$T_L(f,g) := L(f \cdot g) - L(f) \cdot L(g)$$

and give some new results.

Remark 2.1.5. We will use the terminology "Chebyshev-Grüss inequalities", referring to Grüss inequalities for (special cases of) generalized Chebyshev functionals. These inequalities have the general form

$$|T_L(f,g)| \leq E(L,f,g),$$

where E is an expression in terms of certain properties of L and some kind of oscillations of f and g.

2.1.1 Results on compact intervals of the real axis

Another result we recall is a special form of a theorem given by D. Andrica and C. Badea (see [10]):

Theorem 2.1.6. *Let $I := [a,b]$ be a compact interval of the real axis, $B(I)$ the space of real-valued and bounded functions defined on I, and L a positive linear functional satisfying $L(e_0) = 1$ where $e_0 : I \ni x \mapsto 1$. Assuming that for $f,g \in B(I)$ one has $m \leq f(x) \leq M$, $p \leq g(x) \leq P$ for all $x \in I$, the following holds:*

$$|T_L(f,g)| \leq \frac{1}{4}(M-m)(P-p). \tag{2.1.2}$$

Remark 2.1.7. Note that the positive linear functional is not present on the right hand side of the estimate.

The following pre-Chebyshev-Grüss inequality was given by A. Mc. D. Mercer and P. R. Mercer (see [80]) in 2004.

Theorem 2.1.8. *For a positive linear functional $L : B(I) \to \mathbb{R}$, with $L(e_0) = 1$, one has:*

$$|T_L(f,g)| \leq \frac{1}{2}\min\{(M-m)L\left(|g-G|\right), (P-p)L\left(|f-F|\right)\}$$

where $m \leq f(x) \leq M$, $p \leq g(x) \leq P$ for all $x \in I$, $F := Lf$ and $G := Lg$.

Remark 2.1.9. This is a more adequate result than (2.1.2) because the positive linear functional appears on both the left and the right hand side of the inequality.

Using the least concave majorant of the modulus of continuity, the authors of [2] obtained a Grüss inequality for the functional $L(f) = H(f;x)$, where $H : C[a,b] \to C[a,b]$ is a positive linear operator and $x \in [a,b]$ is fixed. B. Gavrea and I. Gavrea [40] were the first to observe the possibility of using moduli in this context.

For $X = [a,b]$ the following weak inequality was given in [2]. It shows how non-multiplicative the functional $L(f)$ is, for a given $x \in [a,b]$.

Theorem 2.1.10. *If $f,g \in C[a,b]$ and $x \in [a,b]$ fixed, then the inequality*

$$|T(f,g;x)| \le \frac{1}{4}\widetilde{\omega}\left(f;2\cdot\sqrt{2H((e_1-x)^2;x)}\right)\cdot\widetilde{\omega}\left(g;2\cdot\sqrt{2H((e_1-x)^2;x)}\right)$$

holds, where $T(f,g;x) := H(f\cdot g;x) - H(f;x)\cdot H(g;x)$.

This was the first result in this setting, which we have further improved. In this sense, we recall the following remark that was also given in [2].

Remark 2.1.11. The above result can be remarkably generalized by replacing $([a,b],|\cdot|)$ by a compact metric space (X,d), $H((e_1-x)^2;x)$ by $H(d^2(\cdot,x);x)$, and $K(\cdot,f;C[a,b],C^1[a,b])$ by $K(\cdot,f;C(X),Lip1)$.

Another pre-Chebyshev-Grüss inequality was given in [2] (see the proof of Theorem 3 there).

We have

$$\begin{aligned}&|T(f,g;x)|\\ &\le \frac{1}{2}\cdot\min\{\|f\|_\infty\cdot\widetilde{\omega}(g,4\cdot L_n(|e_1-x|;x));\|g\|_\infty\cdot\widetilde{\omega}(f,4\cdot L_n(|e_1-x|;x))\},\end{aligned} \tag{2.1.3}$$

for $f,g \in C[-1,1]$ and $x \in [-1,1]$ fixed.

These auxiliary results can be applied to positive linear operators, just like in [2]. We will present such applications in the following section.

2.1.2 Applications for positive linear operators

2.1.2.1 Bernstein operator

By taking $H = B_n$ in Theorem 2.1.10, a first Chebyshev-Grüss inequality for the Bernstein operator was obtained (see Remark 2 in [2]).

Theorem 2.1.12. *The Chebyshev-Grüss inequality*

$$\begin{aligned}&|B_n(fg;x)-B_n(f;x)B_n(g;x)|\\ &\le \frac{1}{4}\widetilde{\omega}\left(f;2\sqrt{2B_n((e_1-x)^2;x)}\right)\cdot\widetilde{\omega}\left(g;2\sqrt{2B_n((e_1-x)^2;x)}\right)\\ &= \frac{1}{4}\widetilde{\omega}\left(f;2\sqrt{\frac{2x(1-x)}{n}}\right)\cdot\widetilde{\omega}\left(g;2\sqrt{\frac{2x(1-x)}{n}}\right)\\ &\le \frac{1}{4}\cdot\widetilde{\omega}\left(f;\frac{1}{\sqrt{2n}}\right)\cdot\widetilde{\omega}\left(g;\frac{1}{\sqrt{2n}}\right)\end{aligned}$$

holds, for two functions $f,g \in C[0,1]$.

2.1.2.2 Hermite-Fejér interpolation operator

If in Theorem 2.1.10 one takes $H := H_{2n-1}$, then the following inequality holds.

Theorem 2.1.13. *For two functions $f, g \in C[-1,1]$, the inequality*

$$\begin{aligned} |T(f,g;x)| &= |H_{2n-1}(f \cdot g; x) - H_{2n-1}(f;x) \cdot H_{2n-1}(g;x)| \\ &\leq \frac{1}{4}\widetilde{\omega}\left(f; \frac{2\sqrt{2}}{\sqrt{n}}\,|T_n(x)|\right) \cdot \widetilde{\omega}\left(g; \frac{2\sqrt{2}}{\sqrt{n}}\,|T_n(x)|\right) \end{aligned} \tag{2.1.4}$$

holds.

This result is dissapointing (see Remark 4 in [2]), because H_{2n-1} approximates much faster than Bernstein. This is the reason why another approach was presented, implying a pre-Chebyshev-Grüss inequality applied to our operator.

In the case of the Hermite-Fejér operator, the inequality (2.1.3) looks like:

$$\begin{aligned} &|T(f,g;x)| \\ &\leq \frac{1}{2}\min\left\{\|f\|_\infty \cdot \widetilde{\omega}\left(g; \frac{40 \cdot |T_n(x)| \ln n}{n}\right); \|g\|_\infty \cdot \widetilde{\omega}\left(f; \frac{40 \cdot |T_n(x)| \ln n}{n}\right)\right\}, \end{aligned} \tag{2.1.5}$$

for $f, g \in C[-1,1]$ and $x \in [-1,1]$ fixed (see Theorem 8 in [2]). The first absolute moment of the Hermite-Fejér interpolation operator was used here. In [45] it was proven that the following estimate holds (see also the appendix in [2] for a detailed proof).

$$H_{2n-1}(|e_1 - x|\,; x) \leq \frac{4}{n}\,|T_n(x)|\left(\sqrt{1-x^2}\ln n + 1\right) \leq 10\,|T_n(x)|\,\frac{\ln n}{n},\ n \geq 2.$$

The following remark states an important observation (see Remark 9 in [2]).

Remark 2.1.14. If one of the functions f or g in (2.1.5) is in $Lip1$, we have $|T(f,g;x)| = \mathcal{O}\left(\frac{\ln n}{n}\right)$, $n \to \infty$. The inequality (2.1.4) implies in this case only $|T(f,g;x)| = o\left(\frac{1}{\sqrt{n}}\right)$. Also the relation (2.1.4) implies $|T(f,g;x)| = \mathcal{O}\left(\frac{1}{n}\right)$ for $f, g \in Lip1$. This cannot be concluded from (2.1.5).

2.1.2.3 Convolution-type operators

If in Theorem 2.1.10 one takes the convolution operators with the Fejér-Korovkin kernel $K_{m(n)}$ for $m(n) = n-1$, then it holds (see Remark 6 in [2])

$$\begin{aligned} |T(f,g;x)| &= |G_{n-1}(f \cdot g; x) - G_{n-1}(f;x) \cdot G_{n-1}(g;x)| \\ &\leq \frac{1}{4} \cdot \widetilde{\omega}\left(f; 4\sqrt{2}\frac{\pi}{n+1}\right) \cdot \widetilde{\omega}\left(g; 4\sqrt{2}\frac{\pi}{n+1}\right) \\ &= \mathcal{O}\left(\widetilde{\omega}\left(f; \frac{1}{n}\right) \cdot \widetilde{\omega}\left(g; \frac{1}{n}\right)\right). \end{aligned}$$

This is an improved result with respect to the ones for the Bernstein and the Hermite-Fejér operators.

2.1.3 Results on compact metric spaces

Let $C(X) = C_{\mathbb{R}}(X, d)$ be the Banach lattice of real-valued continuous functions defined on the compact metric space (X, d) and consider positive linear operators $H : C(X) \to C(X)$ reproducing constant functions. For $x \in X$ we take $L = \epsilon_x \circ H$, so $L(f) = H(f; x)$. We are interested in the degree of non-multiplicativity of such operators. Consider two functions $f, g \in C(X)$ and define the bilinear functional

$$T(f, g; x) := H(f \cdot g; x) - H(f; x) \cdot H(g; x),\ x \in X.$$

We recall a pre-Chebyshev-Grüss inequality on a compact metric space which was proven in [2].

For a linear bounded functional $L : C(X) \to \mathbb{R}$, reproducing constant functions, where $C(X)$ is a compact metric space with metric d, there exist positive linear functionals $L_+,\ L_-,\ |L|$ such that the following relations hold

$$L = L_+ - L_- \text{ and } |L| = L_+ + L_-.$$

In the case of positivity of L, the relation $|L| = L_+ = L$ holds. Using the same idea as for the proof of A. Mercer and P. Mercer's inequality in Theorem 2.1.8, because $M - m = \omega(f; d(X))$, $P - p = \omega(g; d(X))$, where $m = \inf f(x)$, $M = \sup f(x)$, $p = \inf g(x)$, $P = \sup g(x)$, the following inequality was proven in [2] (see Theorem 1 there).

Theorem 2.1.15. *Let $L : C(X) \to \mathbb{R}$ be a linear, bounded functional, $L(1) = 1$, defined on a compact metric space $C(X)$. Then the inequality*

$$|L(fg) - L(f)L(g)| \le \frac{1}{2} \cdot \min\{\omega(f; d(X))\,|L|\,(|g - G|), \omega(g; d(X))\,|L|\,(|f - F|)\}$$

holds.

It was also shown through an example that the above inequality is sharp in the sense that a non-positive functional A that reproduces constant functions exists, such that equality occurs.

2.2 Main results

2.2.1 On Chebyshev's Inequality

In this section we generalize Chebyshev's inequalities from Theorem 2.1.1 and first give bounds for $T_L(f, g)$ for continuously differentiable functions. The two results in this section were already published in [57].

Although the proposition below appears to be well-known, we were unable to locate a reference.

Proposition 2.2.1. *Let $L : C[a, b] \to \mathbb{R}$ be a positive linear functional with $L(e_0) = 1$. If $f, g \in C[a, b]$ are both increasing (decreasing) functions, then the inequality*

$$L(f \cdot g) \ge L(f) \cdot L(g)$$

holds.

Proof. From the monotonicity of f and g we have

$$[f(x) - f(y)] \cdot [g(x) - g(y)] \geq 0,$$

for all $x, y \in [a, b]$, i.e.,

$$f(x) \cdot g(x) - f(x) \cdot g(y) - f(y) \cdot g(x) + f(y) \cdot g(y) \geq 0.$$

Applying L with respect to the variable x to this last inequality gives

$$L^x(f \cdot g) - g(y) \cdot L^x(f) - f(y) \cdot L^x(g) + f(y) \cdot g(y) \geq 0,$$

for all $y \in [a, b]$. Here we used that $L(e_0) = 1$. If we now use L with respect to the variable y, we get

$$L^x(f \cdot g) - L^y(g) \cdot L^x(f) - L^y(f) \cdot L^x(g) + L^y(f \cdot g) \geq 0,$$

and this is equivalent to

$$L(f \cdot g) \geq L(f) \cdot L(g).$$

□

Using the above Proposition 2.2.1, we will give a Chebyshev-Grüss inequality for the functional L, so we now prove the following theorem (for a different proof, see Theorem 4 in [2]):

Theorem 2.2.2. *If L given as above is a positive linear functional with $L(e_0) = 1$, then for the bilinear functional*

$$T_L(f, g) := L(f \cdot g) - L(f) \cdot L(g)$$

we have the inequality

$$|T_L(f, g)| \leq \left\| \frac{f'}{h'} \right\|_\infty \cdot \left\| \frac{g'}{h'} \right\|_\infty \cdot |T_L(h, h)|,$$

where $f, g, h \in C^1[a, b]$ and $h'(t) \neq 0$ for each $t \in [a, b]$.

Proof.

We may suppose that $h'(t) > 0$, $t \in [a, b]$. Let $F := \left\| \frac{f'}{h'} \right\|_\infty$, $G := \left\| \frac{g'}{h'} \right\|_\infty$. Then all four functions $Fh \pm f$, $Gh \pm g$ are increasing. According to Chebyshev's inequality, we have

$$L[(Fh + f) \cdot (Gh + g)] \geq L(Fh + f) \cdot L(Gh + g),$$
$$L[(Fh - f) \cdot (Gh - g)] \geq L(Fh - f) \cdot L(Gh - g).$$

Adding these two inequalities yields

$$\begin{aligned}
&L[(Fh + f) \cdot (Gh + g)] + L[(Fh - f) \cdot (Gh - g)] \\
&\geq L(Fh + f) \cdot L(Gh + g) + L(Fh - f) \cdot L(Gh - g) \\
&\Longleftrightarrow L(FGh^2 + Fgh + Gfh + fg) + L(FGh^2 - Fgh - Gfh + fg) \\
&\geq [L(Fh) + L(f)] \cdot [L(Gh) + L(g)] + [L(Fh) - L(f)] \cdot [L(Gh) - L(g)] \\
&\Longleftrightarrow FG \cdot L(h^2) + F \cdot L(gh) + G \cdot L(fh) + L(fg) + FG \cdot L(h^2) \\
&- F \cdot L(gh) - G \cdot L(fh) + L(fg) \\
&\geq FG \cdot [L(h)]^2 + L(Fh) \cdot L(g) + L(f) \cdot L(Gh) + L(f) \cdot L(g) + FG \cdot [L(h)]^2 \\
&- L(Fh) \cdot L(g) - L(f) \cdot L(Gh) + L(f) \cdot L(g) \\
&\Longleftrightarrow 2 \cdot FG \cdot L(h^2) + 2 \cdot L(f \cdot g) \geq 2 \cdot FG \cdot [L(h)]^2 + 2 \cdot L(f) \cdot L(g)
\end{aligned}$$

and dividing both sides by 2, we get

$$FG \cdot [L(h^2) - (Lh)^2] \geq (Lf) \cdot (Lg) - L(f \cdot g). \tag{2.2.1}$$

Changing now g by $-g$ in (2.2.1) yields

$$FG \cdot [L(h^2) - (Lh)^2] \geq -(Lf) \cdot (Lg) + L(f \cdot g). \tag{2.2.2}$$

From (2.2.1) and (2.2.2) we derive

$$|L(f \cdot g) - (Lf) \cdot (Lg)| \leq FG \cdot (L(h^2) - (Lh)^2),$$

i.e.,

$$|T_L(f,g)| \leq \left\| \frac{f'}{h'} \right\|_\infty \cdot \left\| \frac{g'}{h'} \right\|_\infty \cdot |T_L(h,h)|,$$

and this is the desired inequality. □

2.2.2 (Pre-) Chebyshev-Grüss inequalities on compact intervals of the real axis

For $X = [a,b]$, we give a sligthly better result that improves Theorem 2.1.10, by removing the constant $\sqrt{2}$ in the arguments of the least concave majorants. This proof was already presented in [100] (see Theorem 4.1. there). Again, the interest is in the degree of non-multiplicativity of a positive linear operator $H : C[a,b] \to C[a,b]$ reproducing constant functions. We state and prove the following:

Theorem 2.2.3. *If $f, g \in C[a,b]$ and $x \in [a,b]$ is fixed, then the inequality*

$$|T(f,g;x)| \leq \frac{1}{4}\tilde{\omega}\left(f; 2\sqrt{H((e_1-x)^2;x)}\right) \cdot \tilde{\omega}\left(g; 2\sqrt{H((e_1-x)^2;x)}\right) \tag{2.2.3}$$

holds. The constant $\frac{1}{4}$ is sharp, i.e., there exist non-trivial pairs of functions f and g, such that equality holds.

Proof. Let $f, g \in C[a,b]$ and $r, s \in C^1[a,b]$. Just like in the proof of Theorem 2.1.10 (see proof of Theorem 2 in [2]), we use the Cauchy-Schwarz inequality for positive linear functionals:

$$|H(f;x)| \leq H(|f|;x) \leq \sqrt{H(f^2;x) \cdot H(1;x)} = \sqrt{H(f^2;x)},$$

so we have

$$T(f,f;x) = H(f^2;x) - H(f;x)^2 \geq 0.$$

Thus T is a positive bilinear form on $C[a,b]$. Using Cauchy-Schwarz for T, we obtain

$$|T(f,g;x)| \leq \sqrt{T(f,f;x)T(g,g;x)} \leq \|f\|_\infty \cdot \|g\|_\infty.$$

As stated before, $H : C[a,b] \to C[a,b]$ is a positive linear operator that reproduces constant functions, so that $H(\cdot;x)$, with fixed $x \in [a,b]$, is a positive linear functional that can be represented as

$$H(f;x) = \int_a^b f(t)d\mu_x(t),$$

where μ_x is a probability measure on $[a,b]$, i.e., $\int_a^b d\mu_x(t) = 1$. The interest is in finding an upper bound for the following:

$$\begin{aligned} |T(f,g;x)| &= |T(f-r+r,g-s+s;x)| \\ &\le |T(f-r,g-s;x)| + |T(f-r,s;x)| + |T(r,g-s;x)| + |T(r,s;x)| . \end{aligned}$$

What is different from the proof of Theorem 2.1.10 is that we replace a part of the proof with the following results. We first consider Theorem 4 from the same paper [2]. Let the function h in this theorem be equal to e_1. Then we can write

$$|T(r,s;x)| \le \|r'\|_\infty \cdot \|s'\|_\infty \cdot |T(e_1,e_1;x)|$$

and we know that

$$0 \le T(e_1,e_1;x) = H(e_2;x) - H(e_1;x)^2 \le H((e_1-x)^2;x).$$

This last inequality is true, because

$$\begin{aligned} H((e_1-x)^2;x) &= H(e_2 - 2\cdot e_1 \cdot x + x^2;x) \\ &= H(e_2;x) - 2\cdot x \cdot H(e_1;x) + x^2 \cdot H(e_0;x) \\ &\ge H(e_2;x) - H(e_1;x)^2 \end{aligned}$$

is equivalent to

$$x^2 - 2\cdot x \cdot H(e_1;x) + H(e_1;x)^2 = (x - H(e_1;x))^2 \ge 0.$$

We then get

$$|T(r,s;x)| \le \|r'\|_\infty \cdot \|s'\|_\infty \cdot H((e_1-x)^2;x).$$

For $f-r \in C[a,b]$ and $g-s \in C[a,b]$ we have

$$|T(f-r,g-s;x)| \le \|f-r\|_\infty \cdot \|g-s\|_\infty .$$

Moreover, if $f-r \in C[a,b]$ and $s \in C^1[a,b]$, then

$$\begin{aligned} |T(f-r,s;x)| &\le \sqrt{T(f-r,f-r;x)\cdot T(s,s;x)} \\ &\le \|f-r\|_\infty \cdot \|s'\|_\infty \cdot \sqrt{H((e_1-x)^2;x)} \end{aligned}$$

and similarly, for $r \in C^1[a,b]$, $g-s \in C[a,b]$, we obtain

$$|T(r,g-s;x)| \le \|r'\|_\infty \cdot \|g-s\|_\infty \cdot \sqrt{H((e_1-x)^2;x)}.$$

If we combine all these inequalities, we have

$$\begin{aligned} &|T(f,g;x)| \le \|f-r\|_\infty \cdot \|g-s\|_\infty + \|f-r\|_\infty \cdot \|s'\|_\infty \cdot \sqrt{H((e_1-x)^2;x)} \\ &+ \|r'\|_\infty \cdot \|g-s\|_\infty \cdot \sqrt{H((e_1-x)^2;x)} + \|r'\|_\infty \cdot \|s'\|_\infty \cdot H((e_1-x)^2;x) \\ &= \|f-r\|_\infty \cdot \left\{ \|g-s\|_\infty + \|s'\|_\infty \cdot \sqrt{H((e_1-x)^2;x)} \right\} \\ &+ \|r'\|_\infty \cdot \sqrt{H((e_1-x)^2;x)} \cdot \left\{ \|g-s\|_\infty + \|s'\|_\infty \cdot \sqrt{H((e_1-x)^2;x)} \right\} \\ &= \left\{ \|f-r\|_\infty + \|r'\|_\infty \cdot \sqrt{H((e_1-x)^2;x)} \right\} \\ &\cdot \left\{ \|g-s\|_\infty + \|s'\|_\infty \cdot \sqrt{H((e_1-x)^2;x)} \right\} . \end{aligned}$$

We now pass to the infimum with respect to each of r, s and we obtain the wanted result:

$$\begin{aligned}
&|T(f,g;x)| \\
&\leq K\left(\sqrt{H((e_1-x)^2;x)}, f; C^0, C^1\right) \cdot K\left(\sqrt{H((e_1-x)^2;x)}, g; C^0, C^1\right) \\
&= \frac{1}{2}\widetilde{\omega}\left(f; 2\sqrt{H((e_1-x)^2;x)}\right) \cdot \frac{1}{2}\widetilde{\omega}\left(g; 2\sqrt{H((e_1-x)^2;x)}\right) \\
&= \frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{H((e_1-x)^2;x)}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{H((e_1-x)^2;x)}\right).
\end{aligned}$$

This ends our proof. □

The following two remarks and the example proving that inequality (2.2.3) is sharp can also be found in [58].

Remark 2.2.4. Here the moduli of continuity are oscillations defined with respect to functions f on the whole domain $X = [a, b]$. In order to improve some results, we will later propose a new approach, in which the oscillations are related to the support of the involved functional.

Remark 2.2.5. The inequality (2.2.3) is sharp in the sense that a positive linear operator reproducing constant and linear functions and functions $f, g \in C[a, b]$ exist such that equality occurs.

Example 2.2.6. Consider $f = g := e_1$. Then we have

$$\omega(f;t) = \omega(e_1;t) = \sup\{|x-y| : |x-y| \leq t\} = t.$$

Since $\omega(f;\cdot)$ is linear, we get $\widetilde{\omega}(f;\cdot) = \omega(f;\cdot)$. The left-hand side in Theorem 2.2.3 is

$$|T(f,g;x)| = H(e_2;x) - (H(e_1;x))^2$$

and the right-hand side is

$$\begin{aligned}
&\frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{H((e_1-x)^2;x)}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{H((e_1-x)^2;x)}\right) \\
&= \frac{1}{4} \cdot \left(2\sqrt{H((e_1-x)^2;x)}\right)^2 = H((e_1-x)^2;x).
\end{aligned}$$

By choosing a positive linear operator $H : C[a,b] \to [a,b]$ such that $He_0 = e_0$ and $He_1 = e_1$, we get

$$\begin{aligned}
H((e_1-x)^2;x) &= H(e_2 - 2xe_1 + x^2;x) \\
&= H(e_2;x) - 2xH(e_1;x) + x^2 = H(e_2;x) - x^2 \\
&= H(e_2;x) - (H(e_1;x))^2,
\end{aligned}$$

so we obtain equality between the two sides.

2.2.2.1 A Chebyshev-Grüss-type inequality involving differences $T_L(e_1, e_1)$

In this subsection, we introduce some results already published in [57].

Next we will give a new upper bound for $|T_L(f,g)|$ involving $\widetilde{\omega}$.

Theorem 2.2.7. *If $L : C[a,b] \to \mathbb{R}$ is a positive linear functional with $L(e_0) = 1$, then for $f, g \in C[a,b]$ we have*

$$|T_L(f,g)| \leq \frac{1}{4} \cdot \widetilde{\omega}\left(f; 2 \cdot \sqrt{T_L(e_1,e_1)}\right) \cdot \widetilde{\omega}\left(g; 2 \cdot \sqrt{T_L(e_1,e_1)}\right),$$

where $\widetilde{\omega}$ is the least concave majorant of the modulus of continuity, ω, and

$$T_L(e_1, e_1) = L(e_2) - [L(e_1)]^2.$$

Moreover,

$$T_L\left(\frac{e_1 - a}{b-a}, \frac{e_1 - a}{b-a}\right) \leq \frac{1}{4},$$

with equality holding if and only if $L = \frac{1}{2} \cdot (\epsilon_a + \epsilon_b)$, where $\epsilon_x(f) = f(x)$, $x \in \{a, b\}$.

Proof. First we use the Cauchy-Schwarz inequality for positive linear functionals:

$$L(f) \leq L(|f|) \leq \sqrt{L(f^2) \cdot L(e_0)} = \sqrt{L(f^2)},$$

so we have

$$T_L(f,f) = L(f^2) - L(f)^2 \geq 0,$$

for all $f \in C[a,b]$. Hence, T_L is a positive bilinear form on $C[a,b]$. Using Cauchy-Schwarz for T_L, for all $f, g \in C[a,b]$ we get

$$|T_L(f,g)| \leq \sqrt{T_L(f,f) \cdot T_L(g,g)} \leq \|f\|_\infty \cdot \|g\|_\infty.$$

For $f, g \in C[a,b]$ fixed and $r, s \in C^1[a,b]$ arbitrary, we decompose as follows:

$$\begin{aligned}
&|T_L(f,g)| = |T_L(f - r + r, g - s + s)| \\
&\leq |T_L(f-r, g-s)| + |T_L(f-r, s)| + |T_L(r, g-s)| + |T_L(r,s)|.
\end{aligned}$$

Now $f - r, g - s \in C[a,b]$, so that

$$|T_L(f-r, g-s)| \leq \|f - r\|_\infty \cdot \|g - s\|_\infty.$$

For the second summand we have

$$\begin{aligned}
|T_L(f-r,s)| &\leq \sqrt{T_L(f-r, f-r) \cdot T_L(s,s)} \\
&\leq \|f-r\|_\infty \cdot \sqrt{T_L(s,s)} \\
&\leq \|f-r\|_\infty \cdot \|s'\|_\infty \cdot \sqrt{T_L(e_1,e_1)},
\end{aligned}$$

where the last step follows from Theorem 2.2.2 with $f = g = s$ and $h = e_1$. Likewise,

$$|T_L(r, g-s)| \leq \|g-s\|_\infty \cdot \|r'\|_\infty \cdot \sqrt{T_L(e_1,e_1)}.$$

Finally,

$$|T_L(r,s)| \le \sqrt{T_L(r,r)\cdot T_L(s,s)} \le \|r'\|_\infty \cdot \|s'\|_\infty \cdot T_L(e_1,e_1),$$

by taking $f = r$, $g = s$ and $h = e_1$ in Theorem 2.2.2. Hence,

$$\begin{aligned}
&|T_L(f,g)| \\
&\le \|f-r\|_\infty \cdot \|g-s\|_\infty + \|f-r\|_\infty \cdot \|s'\|_\infty \cdot \sqrt{T_L(e_1,e_1)} \\
&+ \|g-s\|_\infty \cdot \|r'\|_\infty \cdot \sqrt{T_L(e_1,e_1)} + \|r'\|_\infty \cdot \|s'\|_\infty \cdot T_L(e_1,e_1) \\
&\le \left(\|f-r\|_\infty + \|r'\|_\infty \cdot \sqrt{T_L(e_1,e_1)}\right) \cdot \left(\|g-s\|_\infty + \|s'\|_\infty \cdot \sqrt{T_L(e_1,e_1)}\right).
\end{aligned}$$

Passing to the infimum over r and s yields

$$\begin{aligned}
|T_L(f,g)| &\le K(\sqrt{T_L(e_1,e_1)}, f; C^0, C^1) \cdot K(\sqrt{T_L(e_1,e_1)}, g; C^0, C^1) \\
&= \frac{1}{4} \cdot \widetilde{\omega}(f; 2\cdot\sqrt{T_L(e_1,e_1)}) \cdot \widetilde{\omega}(g; 2\cdot\sqrt{T_L(e_1,e_1)}).
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
T_L\left(\frac{e_1-a}{b-a}, \frac{e_1-a}{b-a}\right) &= L\left(\left(\frac{e_1-a}{b-a}\right)^2\right) - L\left(\frac{e_1-a}{b-a}\right) \cdot L\left(\frac{e_1-a}{b-a}\right) \\
&\le L\left(\frac{e_1-a}{b-a}\right) - L\left(\frac{e_1-a}{b-a}\right) \cdot L\left(\frac{e_1-a}{b-a}\right) \\
&= L\left(\frac{e_1-a}{b-a}\right) \cdot \left[1 - L\left(\frac{e_1-a}{b-a}\right)\right] \\
&\le \frac{1}{4}
\end{aligned}$$

since $0 \le L\left(\frac{e_1-a}{b-a}\right) \le 1$. Equality holds if and only if

$$L\left(\frac{e_1-a}{b-a}\right) = \frac{1}{2}.$$

Clearly, if $L\left(\left(\frac{e_1-a}{b-a}\right)^2\right) = L\left(\frac{e_1-a}{b-a}\right) = \frac{1}{2}$, then

$$T_L\left(\frac{e_1-a}{b-a}, \frac{e_1-a}{b-a}\right) = \frac{1}{4}.$$

Assume now that the latter inequality holds and that

$$L\left(\left(\frac{e_1-a}{b-a}\right)^2\right) < L\left(\frac{e_1-a}{b-a}\right).$$

Then

$$\begin{aligned}
\frac{1}{4} &= T_L\left(\frac{e_1-a}{b-a}, \frac{e_1-a}{b-a}\right) \\
&< L\left(\frac{e_1-a}{b-a}\right) \cdot \left[1 - L\left(\frac{e_1-a}{b-a}\right)\right] \le \frac{1}{4},
\end{aligned}$$

which is a contradiction. Thus

$$L\left(\left(\frac{e_1-a}{b-a}\right)^2\right) = L\left(\frac{e_1-a}{b-a}\right), \text{ or}$$
$$L\left(\frac{e_1-a}{b-a}\right) - \left[L\left(\frac{e_1-a}{b-a}\right)\right]^2 = \frac{1}{4}$$
$$\iff L\left(\frac{e_1-a}{b-a}\right) = \frac{1}{2}.$$

Now, if $L = \frac{1}{2}\cdot(\epsilon_a+\epsilon_b)$, then

$$L\left(\left(\frac{e_1-a}{b-a}\right)^2\right) - \left[L\left(\frac{e_1-a}{b-a}\right)\right]^2$$
$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4} = T\left(\frac{e_1-a}{b-a}, \frac{e_1-a}{b-a}\right).$$

If the latter inequality holds, then we saw above that

$$L\left(\left(\frac{e_1-a}{b-a}\right)^2\right) = L\left(\frac{e_1-a}{b-a}\right),$$

which is equivalent to

$$L\left(\frac{e_1-a}{b-a} - \left(\frac{e_1-a}{b-a}\right)^2\right) = 0.$$

The function in the argument is strictly positive in (a,b). So the above inequality is equivalent to L being supported by $\{a,b\}$, i.e.,

$$L = \alpha\cdot\epsilon_a + (1-\alpha)\cdot\epsilon_b,$$

for some $\alpha\in[0,1]$. On the other hand,

$$L\left(\left(\frac{e_1-a}{b-a}\right)^2\right) = L\left(\frac{e_1-a}{b-a}\right) = (1-\alpha) = \frac{1}{2},$$

and so $\alpha = \frac{1}{2}$ and $L = \frac{1}{2}\cdot(\epsilon_a+\epsilon_b)$. This concludes the proof. □

Example 2.2.8. If the functional $L : C[a,b]\to\mathbb{R}$ is again given by

$$L(f) = \frac{1}{b-a}\int_a^b f(x)dx,$$

then the inequality in Theorem 2.2.7 holds with

$$T_L(e_1,e_1) = \frac{1}{b-a}\int_a^b x^2dx - \frac{1}{(b-a)^2}\left(\int_a^b xdx\right)^2 = \frac{(b-a)^2}{12}.$$

This means that

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\cdot g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\cdot\int_a^b g(x)dx\right| \\
&\leq \frac{1}{4}\cdot\widetilde{\omega}\left(f;\frac{2(b-a)}{\sqrt{12}}\right)\cdot\widetilde{\omega}\left(g;\frac{2(b-a)}{\sqrt{12}}\right) \\
&=\leq \frac{1}{4}\cdot\widetilde{\omega}\left(f;\frac{b-a}{\sqrt{3}}\right)\cdot\widetilde{\omega}\left(g;\frac{b-a}{\sqrt{3}}\right)\ f,g\in C[a,b].
\end{aligned}
$$

If f is absolutely continuous with $f'\in L_\infty([a,b])$, then for any difference $|f(x)-f(y)|$, $y\leq x$, figuring in the definition of $\omega(f;t)$ we observe that

$$
\begin{aligned}
|f(x)-f(y)| &= \left|\int_a^x f'(t)dt - \int_a^y f'(t)dt\right| \\
&= \left|\int_y^x f'(t)dt\right| \leq \int_y^x |f'(t)|\,dt \\
&\leq \|f'\|_{L_\infty([a,b])}\cdot(x-y) \\
&\leq \|f'\|_{L_\infty([a,b])}\cdot t.
\end{aligned}
$$

As a consequence, for any expression figuring in the sup defining $\widetilde{\omega}(f;t)$ we have, for $x<y$,

$$
\begin{aligned}
&\frac{(t-x)\cdot\omega(f;y)+(y-t)\cdot\omega(f;x)}{y-x} \\
&\leq \frac{(t-x)\cdot\|f'\|_{L_\infty([a,b])}\cdot y+(y-t)\cdot\|f'\|_{L_\infty([a,b])}\cdot x}{y-x} \\
&= \|f'\|_{L_\infty([a,b])}\cdot t.
\end{aligned}
$$

So, for $f',g'\in L_\infty([a,b])$ we obtain

$$
\begin{aligned}
&\frac{1}{4}\cdot\widetilde{\omega}\left(f;\frac{2(b-a)}{\sqrt{12}}\right)\cdot\widetilde{\omega}\left(g;\frac{2(b-a)}{\sqrt{12}}\right) \\
&\leq \frac{1}{4}\cdot\frac{4(b-a)^2}{12}\cdot\|f'\|_{L_\infty}\cdot\|g'\|_{L_\infty} \\
&= \frac{(b-a)^2}{12}\cdot\|f'\|_{L_\infty}\cdot\|g'\|_{L_\infty}.
\end{aligned}
$$

Hence our result from Theorem 2.2.7 is best possible since we rediscovered the Chebyshev-Grüss inequality for the integration functional in which the constant $\frac{(b-a)^2}{12}$ is best possible.

We will now give an immediate corollary of Theorem 2.2.7 (see Corollary 5.1. in [57]) in which we replace the second moments $H((e_1-x)^2;x)$ in Theorem 2.2.3 by the smaller quantity $H(e_2;x)-H(e_1;x)^2$, proving that the choice involving second moments is not the ideal one. Nevertheless, the order of approximation is as bad as in inequality (2.1.4) (see the application for the Hermite-Fejér interpolation operator).

Corollary 2.2.9. *If $H : C[a,b] \to C[a,b]$ is a positive linear operator which reproduces constant functions, then for $f, g \in C[a,b]$ and $x \in [a,b]$ fixed we have the inequalities:*

$$\begin{aligned}
&|T(f,g;x)| = |H(f \cdot g;x) - H(f;x) \cdot H(g;x)| \\
&\leq \frac{1}{4} \cdot \tilde{\omega}\left(f; 2 \cdot \sqrt{H(e_2;x) - H(e_1;x)^2}\right) \cdot \tilde{\omega}\left(g; 2 \cdot \sqrt{H(e_2;x) - H(e_1;x)^2}\right) \\
&\leq \frac{1}{4} \cdot \tilde{\omega}\left(f; 2 \cdot \sqrt{H((e_1 - x)^2;x)}\right) \cdot \tilde{\omega}\left(f; 2 \cdot \sqrt{H((e_1 - x)^2;x)}\right).
\end{aligned}$$

Proof. Clearly

$$\begin{aligned}
T(e_1, e_1; x) &= H(e_2;x) - H(e_1;x)^2 \\
&\leq H(e_2;x) - 2 \cdot x \cdot H(e_1;x) + x^2 \cdot H(e_0;x) \\
&= H((e_1 - x)^2;x),
\end{aligned}$$

and the statement in the corollary follows from the monotonicity of $\tilde{\omega}$ with respect to the real variable. □

Remark 2.2.10. If H reproduces linear functions, we have no improvement of Theorem 2.2.3. If on the other hand $H(e_1;x) \neq x$, then the inequality in Corollary 2.2.9 is a better estimate. We will apply the above corollary to some positive operators that don't reproduce linear functions, like the classical Hermite-Fejér, the quasi and the almost Hermite-Fejér interpolation operators, convolution operators and the BLaC operator.

2.2.3 Applications for (positive) linear operators

2.2.3.1 Bernstein operators

By letting $H = B_n$ in Theorem 2.2.3, the improved Chebyshev-Grüss inequality for the Bernstein operator looks as follows:

Theorem 2.2.11. *For two functions $f, g \in C[0,1]$ and $x \in [0,1]$ fixed we have*

$$\begin{aligned}
&|B_n(f \cdot g)(x) - B_n f(x) \cdot B_n g(x)| \\
&\leq \frac{1}{4} \cdot \tilde{\omega}\left(f; 2 \cdot \sqrt{\frac{x(1-x)}{n}}\right) \cdot \tilde{\omega}\left(g; 2 \cdot \sqrt{\frac{x(1-x)}{n}}\right),
\end{aligned} \tag{2.2.4}$$

which implies

$$|B_n(f \cdot g)(x) - B_n f(x) \cdot B_n g(x)| \leq \frac{1}{4} \cdot \tilde{\omega}\left(f; \frac{1}{\sqrt{n}}\right) \cdot \tilde{\omega}\left(g; \frac{1}{\sqrt{n}}\right). \tag{2.2.5}$$

The above application for the Bernstein operator was already given as a remark in [100] (see Remark 5.1). The following remark can also be found in [58].

Remark 2.2.12. In equation (2.2.4), the right-hand side depends on x and vanishes when $x \to 0$ or $x \to 1$. The maximum value of it, as a function of x, is attained for $x = \frac{1}{2}$, as (2.2.5) illustrates. On the other hand, we observe that in (2.2.4) the oscillations, expressed in terms of $\tilde{\omega}$, are relative to the whole interval $[0,1]$.

2.2.3.2 King operators

Theorem 2.2.13. *If we take $H = V_n$ in Theorem 2.2.3, then we obtain the following result.*

$$\begin{aligned}
&|V_n(fg;x) - V_n(f;x)V_n(g;x)| \\
&\leq \frac{1}{4} \cdot \widetilde{\omega}\left(f; 2\sqrt{V_n((e_1-x)^2;x)}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{V_n((e_1-x)^2;x)}\right),
\end{aligned}$$

where in the general case the second moment of the V_n operator is given by:

$$\begin{aligned}
V_n((e_1-x)^2;x) &= \frac{r_n(x)}{n} + \frac{n-1}{n}(r_n(x))^2 - 2xr_n(x) + x^2 \\
&= \frac{1}{n} r_n(x)[1 - r_n(x)] + [r_n(x) - x]^2,
\end{aligned}$$

for $0 \leq r_n(x) \leq 1$ and $f, g \in C[0,1]$ continuous functions.

The following result and the remark that follows were already given in [58].

Theorem 2.2.14. *If we consider the operator V_n^*, that reproduces not only constant functions but also e_2, we obtain the inequality*

$$\begin{aligned}
&|T(f,g;x)| = |V_n^*(fg;x) - V_n^*(f;x)V_n^*(g;x)| \\
&\leq \frac{1}{4} \cdot \widetilde{\omega}\left(f; 2\sqrt{V_n^*((e_1-x)^2;x)}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{V_n^*((e_1-x)^2;x)}\right) \\
&= \frac{1}{4} \cdot \widetilde{\omega}\left(f; 2\sqrt{2x(x - V_n^*(e_1;x))}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{2x(x - V_n^*(e_1;x))}\right) \\
&\leq \frac{1}{4} \cdot \widetilde{\omega}\left(f; 2\sqrt{\frac{x(1-x)}{n}}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{\frac{x(1-x)}{n}}\right),
\end{aligned}$$

for $H(e_1;x) = V_n^*(e_1;x) = r_n^*(x)$ and two functions $f, g \in C[0,1]$.

Remark 2.2.15. In this case, one can see that the order of approximation of $V_n(f;x)$ is as good as that of the Bernstein polynomials, so the order of approximation of the Chebyshev-Grüss inequality for V_n^* is also as good as in the case of using the Bernstein operator.

If we consider r_n^{min} and the minimal second moment, we get the following results.

Theorem 2.2.16. *The following Chebyshev-Grüss inequalities for the case of $H = V_n^{\min}$, the positive linear operator that only reproduces constant functions and has minimal second moments, are given as follows:*

$$\begin{aligned}
&|T(f,g;x)| = \left|V_n^{\min}(fg;x) - V_n^{\min}(f;x)V_n^{\min}(g;x)\right| \\
&\leq \frac{1}{4} \cdot \widetilde{\omega}\left(f; 2\sqrt{V_n^{\min}((e_1-x)^2;x)}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{V_n^{\min}((e_1-x)^2;x)}\right) \\
&\leq \begin{cases}
\frac{1}{4} \cdot \widetilde{\omega}(f; 2x) \cdot \widetilde{\omega}(g; 2x) & \text{, for } x \in \left[0, \frac{1}{n}\right) \\
\frac{1}{4} \cdot \cdot \widetilde{\omega}\left(f; 2\sqrt{\frac{1}{n-1}[x(1-x) - \frac{1}{4n}]}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{\frac{1}{n-1}[x(1-x) - \frac{1}{4n}]}\right) & \text{, for } x \in \left[\frac{1}{2n}, 1 - \frac{1}{2n}\right] \\
\frac{1}{4} \cdot \widetilde{\omega}(f; 2(1-x)) \cdot \widetilde{\omega}(g; 2(1-x)) & \text{, for } x \in \left[1 - \frac{1}{2n}, 1\right]
\end{cases}
\end{aligned}$$

for two functions $f, g \in C[0,1]$.

2.2.3.3 Hermite-Fejér operators

For $H = H_{2n-1}$ in Theorem 2.2.3, we get a slightly improved inequality.

Theorem 2.2.17. *If $f, g \in C[-1,1]$ and $x \in [-1,1]$ is fixed, then the inequality*

$$|T(f,g;x)| \leq \frac{1}{4}\widetilde{\omega}\left(f; \frac{2}{\sqrt{n}} \cdot |T_n(x)|\right) \cdot \widetilde{\omega}\left(g; \frac{2}{\sqrt{n}} \cdot |T_n(x)|\right)$$

holds.

By replacing the second moments by a smaller quantity, we obtain a new Chebyshev-Grüss inequality, that was previously given in [57]. We are applying Corollary 2.2.9 to our operator.

To that end, it was shown in DeVore's book ([31], p.43) that for $x \in [-1,1]$ the following holds:

$$\begin{aligned}
&|H_{2n-1}(e_1 - x; x)| = \left| -\frac{1}{n^2} \cdot (1-x^2) \cdot T_n(x) \cdot T_n'(x) - \frac{1}{n} \cdot x \cdot T_n^2(x) \right| \\
&= \frac{1}{n} \cdot |T_n(x)| \cdot \left| \frac{1}{n} \cdot (1-x^2) \cdot T_n'(x) + x \cdot T_n(x) \right| \\
&= \frac{1}{n} \cdot |T_n(x)| \cdot \left| n^{-1} \cdot (1-x^2) \cdot n \cdot \sin(n \cdot \arccos(x)) \cdot \frac{1}{\sqrt{1-x^2}} + x \cdot \cos(n \cdot \arccos(x)) \right| \\
&= \frac{1}{n} \cdot |T_n(x)| \cdot \left| \sqrt{1-x^2} \cdot \sin(n \cdot \arccos(x)) + x \cdot \cos(n \cdot \arccos(x)) \right| \\
&= \frac{1}{n} \cdot |T_n(x)| \cdot |\sin(\arccos(x)) \cdot \sin(n \cdot \arccos(x)) + \cos(\arccos(x)) \cdot \cos(n \cdot \arccos(x))| \\
&= \frac{1}{n} \cdot |T_n(x)| \cdot |\cos((n-1) \cdot \arccos(x))| \\
&= \frac{1}{n} \cdot |T_n(x)| \cdot |T_{n-1}(x)|.
\end{aligned}$$

Hence

$$[H_{2n-1}(e_1 - x; x)]^2 = \frac{1}{n^2} \cdot T_{n-1}^2(x) \cdot T_n^2(x).$$

So we have

$$\begin{aligned}
T(e_1, e_1; x) = H_{2n-1}((e_1 - x)^2; x) - [H_{2n-1}(e_1 - x; x)]^2 &= \frac{1}{n} \cdot T_n^2(x) - \frac{1}{n^2} \cdot T_n^2(x) \cdot T_{n-1}^2(x) \\
&= \frac{1}{n} \cdot T_n^2(x) \cdot \left[1 - \frac{1}{n} \cdot T_{n-1}^2(x)\right]
\end{aligned}$$

and we arrive at the following result.

Theorem 2.2.18. *For the Hermite-Fejér operator, we have*

$$\begin{aligned}
&|H_{2n-1}(f \cdot g; x) - H_{2n-1}(f; x) \cdot H_{2n-1}(g; x)| \\
&\leq \frac{1}{4} \cdot \widetilde{\omega}\left(f; \frac{2}{\sqrt{n}} \cdot |T_n(x)| \cdot \sqrt{1 - \frac{1}{n} \cdot T_{n-1}^2(x)}\right) \cdot \widetilde{\omega}\left(g; \frac{2}{\sqrt{n}} \cdot |T_n(x)| \cdot \sqrt{1 - \frac{1}{n} \cdot T_{n-1}^2(x)}\right).
\end{aligned}$$

This improves inequality (2.1.4).

2.2.3.4 Quasi-Hermite-Fejér interpolation operator

If we take $H = Q_n$ in Theorem 2.2.3, we have

Theorem 2.2.19. *The inequality*

$$
\begin{aligned}
|T(f,g;x)| &\leq \frac{1}{4}\widetilde{\omega}\left(f;2\sqrt{Q_n((e_1-x)^2;x)}\right)\cdot\widetilde{\omega}\left(g;2\sqrt{Q_n((e_1-x)^2;x)}\right)\\
&= \frac{1}{4}\widetilde{\omega}\left(f;2\sqrt{(1-x^2)\cdot\frac{U_n^2(x)}{n+1}}\right)\cdot\widetilde{\omega}\left(g;2\sqrt{(1-x^2)\cdot\frac{U_n^2(x)}{n+1}}\right)\\
&\leq \frac{1}{4}\widetilde{\omega}\left(f;\frac{2\sqrt{1-x^2}}{\sqrt{n+1}}\cdot|U_n(x)|\right)\cdot\widetilde{\omega}\left(g;\frac{2\sqrt{1-x^2}}{\sqrt{n+1}}\cdot|U_n(x)|\right) \qquad (2.2.6)
\end{aligned}
$$

holds, for $f,g \in C[-1,1]$ and $x \in [-1,1]$.

A pre-Chebyshev-Grüss inequality for this operator can also be given. Applying our interpolation operator to inequality (2.1.3), we get the following result:

Theorem 2.2.20. *The pre-Chebyshev-Grüss inequality*

$$
|T(f,g;x)| \leq \frac{1}{2}\min\{\|f\|_\infty\cdot\widetilde{\omega}\left(g;4\cdot Q_n(|e_1-x|\,;x)\right);\|g\|_\infty\cdot\widetilde{\omega}\left(f;4Q_n(|e_1-x|\,;x)\right)\}
$$

holds.

Proof. We need to estimate the first absolute moments for this operator. This is given by

$$
\begin{aligned}
Q_n(|e_1-x|\,;x) &= |-1-x|\cdot\frac{1-x}{2(n+1)^2}\cdot U_n^2(x) + |1-x|\cdot\frac{1+x}{2(n+1)^2}\cdot U_n^2(x)\\
&\quad+\sum_{v=1}^{n}|x_v-x|\cdot(1-x^2)\cdot\frac{(1-x_v\cdot x)}{(n+1)^2}\cdot\left(\frac{U_n(x)}{x-x_v}\right)^2\\
&=: A+B+C.
\end{aligned}
$$

We first estimate the sum denoted by C.

$$
\begin{aligned}
C :&= (1-x^2)\sum_{v=1}^{n}|x_v-x|\cdot h_v(x)\\
&=: (1-x^2)\sum_{v=1}^{n}W_v(x)\\
&= (1-x^2)\cdot\left(\sum_{v=1}^{j-1}W_v(x)+W_j(x)+\sum_{v=j+1}^{n}W_v(x)\right)\\
&=: (1-x^2)\cdot(I_1+I_2+I_3)\,.
\end{aligned}
$$

For $1 \leq v \leq n$, we have

$$h_v(x) = \frac{(1 - x_v \cdot x) \cdot U_n^2(x)}{(n+1)^2 \cdot (x - x_v)^2}$$
$$= \frac{(1 - x^2) \cdot U_n^2(x)}{(n+1)^2 (x - x_v)^2} + \frac{x \cdot U_n^2(x)}{(n+1)^2 (x - x_v)},$$

so that we get

$$\begin{aligned} W_v(x) &= |x_v - x| \cdot h_v(x) \\ &= |x_v - x| \cdot \left[\frac{(1 - x^2) U_n^2(x)}{(n+1)^2 (x - x_v)^2} + \frac{x U_n^2(x)}{(n+1)^2 (x - x_v)} \right] \\ &\leq \frac{(1 - x^2) U_n^2(x)}{(n+1)^2 \, |x - x_v|} + \frac{|x| \, U_n^2(x)}{(n+1)^2}. \end{aligned}$$

Because the following estimation

$$\frac{(1 - x^2) U_n^2(x)}{(n+1)^2 \, |x - x_v|} \leq \frac{\sqrt{1 - x^2} U_n^2(x) \pi}{(n+1)^2} \cdot \frac{1}{\theta - \theta_v}$$

holds, we obtain

$$\begin{aligned} I_1 + I_3 \leq & \frac{\sqrt{1 - x^2} U_n^2(x) \pi}{(n+1)^2} \cdot \left\{ \sum_{v=1}^{j-1} \frac{1}{|\theta - \theta_v|} + \sum_{v=j+1}^{n} \frac{1}{|\theta - \theta_v|} \right\} \\ & + (n-1) \frac{|x| \, U_n^2(x)}{(n+1)^2}. \end{aligned}$$

In order to estimate the above accolade, we need two cases.

The "left case" implies $\theta_{j-1} < \theta < \theta_j$, for $1 < j < n$ and $v \leq j - 1 (v = j - i)$. Then we have

$$\begin{aligned} \theta - \theta_v &\geq (\theta_{j-1} - \theta_v) + \frac{1}{2} \left(\theta_j - \theta_{j-1} \right) \\ &\geq \frac{(2i - 1)\pi}{2(n+1)}. \end{aligned}$$

For $v \geq j + 1 (v = j + i)$, we get

$$\theta_v - \theta \geq \theta_v - \theta_j = \frac{\pi}{n+1} \cdot i,$$

so in the "left case" we obtain

$$\begin{aligned}
&\sum_{v=1}^{j-1} \frac{1}{|\theta-\theta_v|} + \sum_{v=j+1}^{n} \frac{1}{|\theta-\theta_v|} \\
&\le \sum_{v=1}^{j-1} \frac{2(n+1)}{(2i-1)\pi} + \sum_{v=j+1}^{n} \frac{2(n+1)}{2i\pi} \\
&= 2(n+1)\pi^{-1} \cdot \left[\sum_{v=1}^{j-1} \frac{1}{2v-1} + \sum_{v=1}^{n-j} \frac{1}{2v} \right] \\
&\le 2(n+1)\pi^{-1} \left[1 + \frac{1}{2}\ln(2j-3) + \frac{1}{2}\left(\ln(n-j)+1\right) \right] \\
&\le 2(n+1)\pi^{-1} \left[\frac{3}{2} + \ln\left(\frac{1}{\sqrt{2}} \cdot n \right) \right].
\end{aligned}$$

In the "right case", we obtain analogously

$$\begin{aligned}
&\sum_{v=1}^{j-1} \frac{1}{|\theta-\theta_v|} + \sum_{v=j+1}^{n} \frac{1}{|\theta-\theta_v|} \\
&\le \sum_{v=1}^{j-1} \frac{2(n+1)}{2i\pi} + \sum_{v=j+1}^{n} \frac{2(n+1)}{(2i-1)\pi} \\
&= 2(n+1)\pi^{-1} \cdot \left[\sum_{v=1}^{j-1} \frac{1}{2v} + \sum_{v=1}^{n-j} \frac{1}{2v-1} \right] \\
&\le 2(n+1)\pi^{-1} \left[\frac{3}{2} + \ln\left(\frac{1}{\sqrt{2}} \cdot n \right) \right].
\end{aligned}$$

Taking the results for the two cases together, one gets

$$I_1 + I_3 \le \frac{\sqrt{1-x^2}U_n^2(x)}{n+1} \left(3 + 2\ln\left(\frac{1}{\sqrt{2}}\right) n \right) + \frac{|x|\, U_n^2(x)}{n+1}.$$

For I_2 we have

$$\begin{aligned}
I_2 = W_j(x) &= \left|x_j - x\right| \cdot \frac{(1 - x\cdot x_j)U_n^2(x)}{(n+1)^2(x-x_j)^2} \\
&= \left|\cos\theta_j - \cos\theta\right| \cdot h_j(x) \le \left|\theta_j - \theta\right| \\
&\le \frac{\pi}{2(n+1)} \left|\cos(n+1)\theta\right| \\
&= \frac{\pi}{2(n+1)} \left|T_{n+1}(x)\right|,
\end{aligned}$$

for $x_j = \cos\left(\frac{j}{n+1}\pi\right)$, $1 \le j \le n$.

So, for $n \geq 1$ we have:

$$
\begin{aligned}
I_1 + I_2 + I_3 &\leq \frac{\sqrt{1-x^2}U_n^2(x)}{n+1}\left(3 + 2\ln\left(\frac{1}{\sqrt{2}} \cdot n\right)\right) + \frac{|x|\, U_n^2(x)}{n+1} \\
&+ \frac{\pi}{2(n+1)}\, |T_{n+1}(x)| \\
&\leq \frac{\sqrt{1-x^2}U_n^2(x)}{n+1}\left(2 + 2\ln\left(\frac{1}{\sqrt{2}} \cdot n\right)\right) + \frac{\sqrt{1-x^2}U_n^2(x)}{n+1} \\
&+ \frac{|x|\, U_n^2(x)}{n+1} + \frac{\pi}{2(n+1)} \cdot (U_{n+1}(x) + |x| \cdot U_n(x)) \\
&\leq \frac{|U_n(x)|}{n+1}\left[\sqrt{1-x^2} \cdot \left(2 + 2\ln\left(\frac{1}{\sqrt{2}} \cdot n\right)\right) + 2 + \frac{\pi}{2}\right] \\
&\leq \frac{4\,|U_n(x)|}{n+1}\left[\sqrt{1-x^2} \cdot \ln n + 1\right] \\
&\leq \frac{10\ln(n+1)}{n+1},
\end{aligned}
$$

and from here it follows:

$$
C \leq \frac{(1-x^2) \cdot 10\ln(n+1)}{n+1}.
$$

Then the first absolute value of our operator is

$$
\begin{aligned}
Q_n(|e_1 - x|\,; x) &\leq (1-x^2) \cdot \left\{\frac{U_n^2(x)}{(n+1)^2} + \frac{4\,|U_n(x)|}{n+1}(\sqrt{1-x^2} \cdot \ln n + 1)\right\} \\
&\leq (1-x^2) \cdot \frac{|U_n(x)|}{n+1}\left(1 + 4\left(\sqrt{1-x^2} \cdot \ln n + 1\right)\right) \\
&\leq \frac{10(1-x^2)\,|U_n(x)|\ln(n+1)}{n+1}.
\end{aligned}
$$

By putting this first absolute moment into the above given formula, we get a pre-Chebyshev-Grüss inequality. □

The last result we give here is meaning to give an improvement, by replacing the second moment in inequality (2.2.6) by something smaller.

We need the quantity

$$
\begin{aligned}
T(e_1, e_1; x) &:= Q_n((e_1 - x)^2; x) - [Q_n(e_1 - x; x)]^2 \\
&= (1-x^2) \cdot \frac{U_n^2(x)}{n+1} - \frac{(1-x^2)^2}{(n+1)^2} \cdot U_n^2(x) \cdot [T_{n+1}(x) - x \cdot U_n(x)]^2 \\
&= \frac{1-x^2}{n+1} \cdot U_n^2(x)\left\{1 - \frac{1-x^2}{n+1} \cdot [T_{n+1}(x) - x \cdot U_n(x)]^2\right\} \\
&\leq \frac{1}{n+1}U_n^2(x)\left\{1 - \frac{1-x^2}{n+1}\,[T_{n+1}(x) - x \cdot U_n(x)]^2\right\},
\end{aligned}
$$

so we get

Theorem 2.2.21. *If we take $H = Q_n$ in Corollary 2.2.9, we have*

$$\begin{aligned}|T(f,g;x)| &:= |Q_n(f\cdot g;x) - Q_n(f;x)\cdot Q_n(g;x)|\\ &\le \frac{1}{4}\tilde{\omega}\left(f;\frac{2}{\sqrt{n+1}}|U_n(x)|\cdot\sqrt{1-\frac{1-x^2}{n+1}(T_{n+1}(x)-x\cdot U_n(x))^2}\right)\\ &\cdot\tilde{\omega}\left(g;\frac{2}{\sqrt{n+1}}|U_n(x)|\cdot\sqrt{1-\frac{1-x^2}{n+1}(T_{n+1}(x)-x\cdot U_n(x))^2}\right).\end{aligned}$$

2.2.3.5 Almost-Hermite-Fejér operator

If we take $H = F_{1,0;n}^{\left(\frac{1}{2},-\frac{1}{2}\right)}$ in Theorem 2.2.3, we have

Theorem 2.2.22. *For two functions $f,g \in C[-1,1]$, we have*

$$\begin{aligned}|T(f,g;x)| &\le \frac{1}{4}\tilde{\omega}\left(f;2\sqrt{F_{1,0;n}^{\left(\frac{1}{2},-\frac{1}{2}\right)}((e_1-x)^2;x)}\right)\cdot\tilde{\omega}\left(g;2\sqrt{F_{1,0;n}^{\left(\frac{1}{2},-\frac{1}{2}\right)}((e_1-x)^2;x)}\right)\\ &= \frac{1}{4}\tilde{\omega}\left(f;2\sqrt{\frac{2(1-x)w(x)^2}{3n}}\right)\cdot\tilde{\omega}\left(g;2\sqrt{\frac{2(1-x)w(x)^2}{3n}}\right)\\ &= \frac{1}{4}\tilde{\omega}\left(f;\frac{2\sqrt{2(1-x)}\cdot|w(x)|}{\sqrt{3n}}\right)\cdot\tilde{\omega}\left(g;\frac{2\sqrt{2(1-x)}\cdot|w(x)|}{\sqrt{3n}}\right). \quad (2.2.7)\end{aligned}$$

We also present a pre-Chebyshev-Grüss inequality for this operator, by applying our interpolation operator to inequality (2.1.3).

Theorem 2.2.23. *We have the following pre-Chebyshev-Grüss inequality*

$$\begin{aligned}&|T(f,g;x)|\\ &\le \frac{1}{2}\min\left\{\|f\|_\infty\cdot\tilde{\omega}\left(g;4\cdot F_{1,0;n}^{\left(\frac{1}{2},-\frac{1}{2}\right)}(|e_1-x|;x)\right);\|g\|_\infty\cdot\tilde{\omega}\left(f;4F_{1,0;n}^{\left(\frac{1}{2},-\frac{1}{2}\right)}(|e_1-x|;x)\right)\right\}\\ &\le \frac{1}{2}\min\left\{\|f\|_\infty\cdot\tilde{\omega}\left(g;4c\frac{1+\sqrt{1-x^2}\cdot\ln n}{2n+1}\right);\|g\|_\infty\cdot\tilde{\omega}\left(f;4c\frac{1+\sqrt{1-x^2}\cdot\ln n}{2n+1}\right)\right\},\end{aligned}$$

for $f,g \in C[-1,1]$ and $x \in [-1,1]$ fixed.

We now replace the second moment in inequality (2.2.7) by a smaller quantity.

We have

$$\begin{aligned}&T(e_1,e_1;x) = F_{1,0;n}^{\left(\frac{1}{2},-\frac{1}{2}\right)}((e_1-x)^2;x) - [F_{1,0;n}^{\left(\frac{1}{2},-\frac{1}{2}\right)}(e_1-x;x)]^2\\ &= \frac{2(1-x)\cdot w(x)^2}{3n} - \frac{(1-x)^2\cdot w^2(x)}{(2n+1)^4}\cdot[2(1-x^2)\cdot w'(x)+(2nx-1)\cdot w(x)]^2\\ &= \frac{(1-x)\cdot w^2(x)}{(2n+1)^2}\cdot\left[2n+1-x\cdot\left(1+2\cdot\sum_{k=1}^{n}x_k\right)\right]\\ &- \frac{(1-x)^2\cdot w^2(x)}{(2n+1)^4}\cdot[2(1-x^2)\cdot w'(x)+(2nx-1)\cdot w(x)]^2\\ &= \frac{(1-x)\cdot w(x)^2}{(2n+1)^2}\cdot\left\{2n+1-x\cdot(1+2\cdot\sum_{k=1}^{n}x_k)-\frac{1-x}{(2n+1)^2}[2(1-x^2)w'(x)+(2nx-1)\cdot w(x)]^2\right\}\end{aligned}$$

in order to give a better result in comparison with (2.2.7). Putting

$$h := 2n + 1 - x \cdot (1 + 2 \cdot \sum_{k=1}^{n} x_k) - \frac{1-x}{(2n+1)^2}[2(1-x^2)w'(x) + (2nx - 1) \cdot w(x)]^2,$$

we arrive at the following result from the next theorem, since h is positive.

Theorem 2.2.24. *By taking $H = F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}$ in Corollary 2.2.9, the following inequality*

$$\begin{aligned}|T(f,g;x)| &:= \left|F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}(f \cdot g;x) - F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}(f;x) \cdot F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}(g;x)\right| \\ &\leq \frac{1}{4}\tilde{\omega}\left(f; \frac{2 \cdot |w(x)|}{2n+1} \cdot \sqrt{(1-x) \cdot h}\right) \cdot \tilde{\omega}\left(g; \frac{2 \cdot |w(x)|}{2n+1} \cdot \sqrt{(1-x) \cdot h}\right)\end{aligned}$$

holds.

2.2.3.6 Convolution-type operators

In this section we take into account different degrees $m(n)$, different convolution operators and Chebyshev-Grüss inequalities, respectively.

We now state the following results, already published in [100].

Theorem 2.2.25. *If we consider $f, g \in C[-1,1]$ and the convolution operator of degree $n-1$ with the Fejér-Korovkin kernel in Theorem 2.2.3, we have*

$$\begin{aligned}|T(f,g;x)| &= |G_{n-1}(f \cdot g;x) - G_{n-1}(f;x) \cdot G_{n-1}(g;x)| \\ &\leq \frac{1}{4}\tilde{\omega}\left(f; \frac{4\pi}{n+1}\right) \cdot \tilde{\omega}\left(g; \frac{4\pi}{n+1}\right) \\ &= \mathcal{O}\left(\tilde{\omega}\left(f; \frac{1}{n}\right) \cdot \tilde{\omega}\left(g; \frac{1}{n}\right)\right).\end{aligned}$$

Theorem 2.2.26. *If we consider the convolution operator of degree $2n-2$ with the Jackson kernel in Theorem 2.2.3, we have*

$$\begin{aligned}|T(f,g;x)| &= |G_{2n-2}(f \cdot g;x) - G_{2n-2}(f;x) \cdot G_{2n-2}(g;x)| \\ &\leq \frac{1}{4}\tilde{\omega}\left(f; \frac{2\sqrt{3}}{n}\right) \cdot \tilde{\omega}\left(g; \frac{2\sqrt{3}}{n}\right) \\ &= \mathcal{O}\left(\tilde{\omega}\left(f; \frac{1}{n}\right) \cdot \tilde{\omega}\left(g; \frac{1}{n}\right)\right).\end{aligned}$$

Theorem 2.2.27. *If we consider the convolution operator of degree n with the de La Vallée Poussin kernel in Theorem 2.2.3, we get*

$$\begin{aligned}|T(f,g;x)| &= |G_n(f \cdot g;x) - G_n(f;x) \cdot G_n(g;x)| \\ &\leq \frac{1}{4}\tilde{\omega}\left(f; \frac{2\sqrt{2}}{\sqrt{n+1}}\right) \cdot \tilde{\omega}\left(g; \frac{2\sqrt{2}}{\sqrt{n+1}}\right) \\ &= \mathcal{O}\left(\tilde{\omega}\left(f; \frac{1}{\sqrt{n}}\right) \cdot \tilde{\omega}\left(g; \frac{1}{\sqrt{n}}\right)\right).\end{aligned}$$

This theorem can be slightly improved, if we consider the smaller quantity with respect to the second moment

$$\begin{aligned} T(e_1, e_1; x) &= G_n((e_1 - x)^2; x) - (G_n(e_1; x) - x)^2 \\ &\leq \frac{2}{n+1} - x^2 \cdot \frac{1}{(n+1)^2}. \end{aligned}$$

Taking this into account, we give the following Chebyshev-Grüss inequality for the convolution operator with de La Vallée Poussin kernel:

Theorem 2.2.28. *If we consider the convolution operator with the de La Vallée Poussin kernel and apply Corollary 2.2.9 to it, we have*

$$\begin{aligned} |T(f, g; x)| &= |G_n(f \cdot g; x) - G_n(f; x) \cdot G_n(g; x)| \\ &\leq \frac{1}{4} \cdot \tilde{\omega}\left(f; 2 \cdot \sqrt{T(e_1, e_1; x)}\right) \cdot \tilde{\omega}\left(g; 2 \cdot \sqrt{T(e_1, e_1; x)}\right) \\ &\leq \frac{1}{4} \cdot \tilde{\omega}\left(f; 2 \cdot \frac{\sqrt{2 - \frac{x^2}{n+1}}}{\sqrt{n+1}}\right) \cdot \tilde{\omega}\left(g; 2 \cdot \frac{\sqrt{2 - \frac{x^2}{n+1}}}{\sqrt{n+1}}\right). \end{aligned}$$

The above theorem was already introduced in [57]. The next remark can also be found in [100].

Remark 2.2.29. As we can see, the best degrees of approximation are obtained when dealing with the Chebyshev-Grüss inequality for convolution operators in the cases of Fejér-Korovkin and Jackson kernels.

2.2.3.7 S_{Δ_n}- piecewise linear interpolation operator

In order to obtain a classical Chebyshev-Grüss inequality using S_{Δ_n}, we need the second moment of the operator. We have seen in Section 1.2.10 that the following relationship for the second moments holds:

$$S_{\Delta_n}(e_1 - xe_0)^2(x) \leq \frac{1}{4n^2}.$$

The proof of the previous inequality and the theorem that is given in the sequel have also already been introduced in [58].

By putting $H = S_{\Delta_n}$ in Theorem 2.2.3, the Chebyshev-Grüss inequality for S_{Δ_n} is given by the following result.

Theorem 2.2.30. *If $f, g \in C[0,1]$ and $x \in [0,1]$ is fixed, then the inequality*

$$\begin{aligned} |T(f, g; x)| &\leq \frac{1}{4}\tilde{\omega}\left(f; 2 \cdot \sqrt{S_{\Delta_n}((e_1 - x)^2; x)}\right) \cdot \tilde{\omega}\left(g; 2 \cdot \sqrt{S_{\Delta_n}((e_1 - x)^2; x)}\right) \\ &\leq \frac{1}{4}\tilde{\omega}\left(f; \frac{1}{n}\right) \cdot \tilde{\omega}\left(g; \frac{1}{n}\right) \end{aligned}$$

holds.

2.2.3.8 The BLaC operator

All of the results from this subsection were already published in [101].

If we take $H = BL_n$ in Theorem 2.2.3, we obtain the following result:

Theorem 2.2.31. *For $f, g \in C[0,1]$ and $x \in [0,1]$ fixed, the following inequality*

$$\begin{aligned}
&|T(f,g;x)| \\
&\le \frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{BL_n((e_1-x)^2;x)}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{BL_n((e_1-x)^2;x)}\right) \\
&\le \frac{1}{4}\widetilde{\omega}\left(f; \frac{1}{2^{n-1}}\right) \cdot \widetilde{\omega}\left(g; \frac{1}{2^{n-1}}\right)
\end{aligned}$$

holds.

We can also give a pre-Chebyshev-Grüss inequality, using equation (2.1.3) (see Theorem 6.1 in [100]).

Theorem 2.2.32. *Let $f, g \in C[0,1]$. Then the inequality*

$$\begin{aligned}
&|T(f,g;x)| \\
&\le \frac{1}{2}\min\{\|f\|_\infty \widetilde{\omega}\left(g; 4\cdot BL_n(|e_1-x|;x)\right), \|g\|_\infty \widetilde{\omega}\left(f; 4\cdot BL_n(|e_1-x|;x)\right)\} \\
&\le \frac{1}{2}\min\left\{\|f\|_\infty \widetilde{\omega}\left(g; \frac{1}{2^{n-2}}\right), \|g\|_\infty \widetilde{\omega}\left(f; \frac{1}{2^{n-2}}\right)\right\}
\end{aligned}$$

holds.

An improved Chebyshev-Grüss inequality for the BLaC operator will now be given. In a recent paper [57] we proved that replacing the second moments $H((e_1 - x)^2; x)$ in Theorem 2.2.3 by the smaller quantity $H(e_2; x) - H(e_1; x)^2$ is sometimes a better choice. For the BLaC operator considered in Corollary 2.2.9, the situation is as follows:

We want to estimate the quantity

$$\begin{aligned}
T(e_1, e_1; x) &:= BL_n((e_1-x)^2;x) - [BL_n(e_1-x;x)]^2 \\
&\le \frac{1}{2^{2n}} - [BL_n(e_1-x;x)]^2,
\end{aligned}$$

for all $x \in [0,1]$.

Suppose $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$, for $k \in \{0, \dots, 2^n - 1\}$. This only leaves out the case $x = 1$, when we get $BL_n(e_1 - 1; 1) = 0$. We distinguish between two cases:

Case 1: $x \in \left[\frac{k}{2^n}, \frac{k+\Delta}{2^n}\right)$, for $k \in 0, \dots, 2^n - 1$.

1. First we treat the case $\mathbf{k} = \mathbf{0}$. We have:

$$\begin{aligned}
BL_n(e_1-x;x) &= (\eta_{-1}^n - x)\cdot\varphi_{-1}^n(x) + (\eta_0^n - x)\cdot\varphi_0^n(x) \\
&= \frac{x(1-\Delta)}{2\Delta},
\end{aligned}$$

so we get

$$\begin{aligned}
T(e_1,e_1;x) &\le \frac{1}{2^{2n}} - [BL_n(e_1-x;x)]^2 \\
&= \frac{1}{2^{2n}} - \left[\frac{x(1-\Delta)}{2\cdot\Delta}\right]^2 \\
&= \frac{1}{2^{2n}}\left[1-\left(\frac{2^n\cdot x\cdot(1-\Delta)}{2\cdot\Delta}\right)^2\right] \\
&= \frac{1}{2^{2n}}\left[1-\left(\underbrace{\frac{(A(x)+\Delta+2k)(1-\Delta)}{4\cdot\Delta}}_{(*)}\right)^2\right],
\end{aligned}$$

where we denote $A(x) := 2^{n+1}x - 2k - \Delta$. We need the quantity $(*)$ to be positive. This is the case when $0<\Delta<1$ and $x \in \left(0, \frac{\Delta}{2^n}\right)$, because

$$\begin{aligned}
&A(x)+\Delta+2k \neq 0 \\
\Leftrightarrow\ & A(x) \neq -\Delta-2k \\
\Leftrightarrow\ & x \neq 0
\end{aligned}$$

In the sequel we apply Corollary 2.2.9 to our BLaC operator and we obtain the following result.

Theorem 2.2.33. *For $f,g \in C[0,1]$ and $x \in \left[0,\frac{\Delta}{2^n}\right)$, we have the inequality*

$$\begin{aligned}
|T(f,g;x)| \le \frac{1}{4}\widetilde{\omega}&\left(f;2\cdot\sqrt{\frac{1}{2^{2n}}\left[1-\left(\frac{(A(x)+\Delta+2k)(1-\Delta)}{4\cdot\Delta}\right)^2\right]}\right) \\
&\cdot\widetilde{\omega}\left(g;2\cdot\sqrt{\frac{1}{2^{2n}}\left[1-\left(\frac{(A(x)+\Delta+2k)(1-\Delta)}{4\cdot\Delta}\right)^2\right]}\right).
\end{aligned}$$

This is an estimate better than the one in Theorem 2.2.31 for $0<\Delta<1$ and $x \neq 0$.

2. For $\mathbf{1 \le k \le 2^n - 1}$, we have:

$$\begin{aligned}
BL_n(e_1-x;x) &= \frac{1}{2^{n+1}}\cdot\frac{(1-\Delta)}{\Delta}[2(2^n x - k)-\Delta] \\
BL_n((e_1-x)^2;x) &= (\eta^n_{k-1}-x)^2\cdot\varphi^n_{k-1} + (\eta^n_k - x)^2\cdot\varphi^n_k(x) \\
&\le \frac{1}{2^{2n}},
\end{aligned}$$

so we get

$$\begin{aligned}
T(e_1,e_1;x) &\le \frac{1}{2^{2n}} - [BL_n(e_1-x;x)]^2 \\
&\le \frac{1}{2^{2n}} - \frac{1}{2^{2n+2}}\left[\frac{(1-\Delta)\,[2(2^n\cdot x-k)-\Delta]}{\Delta}\right]^2 \\
&= \frac{1}{2^{2n}}\left[1-\left(\underbrace{\frac{A(x)\cdot(1-\Delta)}{2\cdot\Delta}}_{(**)}\right)^2\right]
\end{aligned}$$

where we denote $A(x) := 2^{n+1}x - 2k - \Delta$. We need the quantity $(**)$ to be positive. This is the case when $0 < \Delta < 1$ and $x \neq \frac{k}{2^n} + \frac{\frac{1}{2}\cdot\Delta}{2^n}$, because

$$\begin{aligned} A(x) &\neq 0 \\ \Leftrightarrow 2^{n+1}x - 2k - \Delta &\neq 0 \\ \Leftrightarrow x &\neq \frac{2k+\Delta}{2^{n+1}}. \end{aligned}$$

We apply Corollary2.2.9 to our BLaC operator to obtain the following

Theorem 2.2.34. *For $f, g \in C[0,1]$ and $x \in \left[\frac{k}{2^n}, \frac{k+\Delta}{2^n}\right)$ with $k \in \{1, \dots, 2^n - 1\}$, we have the inequality*

$$\begin{aligned} |T(f,g;x)| \leq \frac{1}{4}\tilde{\omega}&\left(f; 2\cdot\sqrt{\frac{1}{2^{2n}}\left[1 - \left(\frac{A(x)\cdot(1-\Delta)}{2\cdot\Delta}\right)^2\right]}\right) \\ &\cdot\tilde{\omega}\left(g; 2\cdot\sqrt{\frac{1}{2^{2n}}\left[1 - \left(\frac{A(x)\cdot(1-\Delta)}{2\cdot\Delta}\right)^2\right]}\right), \end{aligned}$$

This is an estimate better than the one in Theorem 2.2.31 for $x \neq \frac{k}{2^n} + \frac{\frac{1}{2}\cdot\Delta}{2^n}$ and $0 < \Delta < 1$.

Case 2: $x \in \left[\frac{k+\Delta}{2^n}, \frac{k+1}{2^n}\right)$, for $k \in 0, \dots, 2^n - 1$. We have:

$$\begin{aligned} BL_n(e_1 - x; x) &= (x - \eta_k^n)\cdot\varphi_k^n(x) \\ &= \frac{2(2^n x - k) - \Delta - 1}{2n+1} \\ BL_n((e_1 - x)^2; x) &= (x - \eta_k^n)^2\cdot\varphi_k^n(x) \\ &\leq \frac{1}{2^{2n}}, \end{aligned}$$

so we get

$$\begin{aligned} T(e_1, e_1; x) &\leq \frac{1}{2^{2n}} - [BL_n(e_1 - x; x)]^2 \\ &\leq \frac{1}{2^{2n}} - \frac{1}{2^{2n+2}}\cdot[A(x) - 1]^2 \\ &= \frac{1}{2^{2n}}\left[1 - \Bigg(\underbrace{\frac{A(x)-1}{2}}_{(***)}\Bigg)^2\right] \end{aligned}$$

where we denote $A(x) := 2^{n+1}x - 2k - \Delta$. We need the quantity $(***)$ to be positive. This is the case when $x \neq \frac{k}{2^n} + \frac{\frac{1}{2}\cdot(1+\Delta)}{2^n}$, because

$$\begin{aligned} A(x) &\neq 1 \\ \Leftrightarrow 2^{n+1}x - 2k - \Delta &\neq 1 \\ \Leftrightarrow x &\neq \frac{2k+\Delta+1}{2^{n+1}}. \end{aligned}$$

Applying Corollary 2.2.9 to the BLaC operator gives

Theorem 2.2.35. *For $f,g \in C[0,1]$ and $x \in \left[\frac{k+\Delta}{2^n}, \frac{k+1}{2^n}\right)$ with $k \in \{0,\ldots,2^n-1\}$, we have the inequality*

$$|T(f,g;x)| \leq \frac{1}{4}\widetilde{\omega}\left(f; 2\cdot\sqrt{\frac{1}{2^{2n}}\left[1-\left(\frac{A(x)-1}{2}\right)^2\right]}\right) \cdot \widetilde{\omega}\left(g; 2\cdot\sqrt{\frac{1}{2^{2n}}\left[1-\left(\frac{A(x)-1}{2}\right)^2\right]}\right).$$

This is an estimate better than the one in Theorem 2.2.31 for $x \neq \frac{k}{2^n} + \frac{\frac{1}{2}\cdot(1+\Delta)}{2^n}$.

2.2.4 (Pre-) Chebyshev-Grüss inequalities on compact metric spaces

In [100] (see Theorem 3.1.) the following was shown, in order to generalize Theorem 2.1.10 in the case of a compact metric space (see Remark 2.1.11).

Theorem 2.2.36. *If $f,g \in C(X)$, where (X,d) is a compact metric space, and $x \in X$, then the inequality*

$$|T(f,g;x)| \leq \frac{1}{4}\widetilde{\omega_d}\left(f; 4\sqrt{H(d^2(\cdot,x);x)}\right) \cdot \widetilde{\omega_d}\left(g; 4\sqrt{H(d^2(\cdot,x);x)}\right)$$

holds, where $H(d^2(\cdot,x);x)$ is the second moment of the positive linear operator H, which reproduces constant functions.

Proof. Let $f,g \in C[a,b]$ and $r,s \in Lip_1$. We use the Cauchy-Schwarz inequality for positive linear functionals:

$$|H(f;x)| \leq H(|f|;x) \leq \sqrt{H(f^2;x)\cdot H(1;x)} = \sqrt{H(f^2;x)},$$

so we have

$$T(f,f;x) = H(f^2;x) - H(f;x)^2 \geq 0.$$

Hence T is a positive bilinear form on $C(X)$. Using the Cauchy-Schwarz inequality for T gives us

$$|T(f,g;x)| \leq \sqrt{T(f,f;x)\cdot T(g,g;x)} \leq \|f\|_\infty \cdot \|g\|_\infty .$$

Because $H : C(X) \to C(X)$ is a positive linear operator reproducing constant functions, $H(f;x)$, with fixed $x \in X$, is a positive linear functional that we can represent as follows

$$H(f;x) := \int_X f(t)\mathrm{d}\mu_x(t),$$

where μ_x is a Borel probability measure on X, i.e., $\int_X \mathrm{d}\mu_x(t) = 1$. For r as above, we have

$$\begin{aligned}
T(r,r;x) &= H(r^2;x) - H(r;x)^2 = \int_X r^2(t)\mathrm{d}\mu_x(t) - \left(\int_X r(u)\mathrm{d}\mu_x(u)\right)^2 \\
&= \int_X \left(r(t) - \int_X r(u)\mathrm{d}\mu_x(u)\right)^2 \mathrm{d}\mu_x(t) \\
&= \int_X \left(\int_X (r(t) - r(u))\,\mathrm{d}\mu_x(u)\right)^2 \mathrm{d}\mu_x(t) \\
&\le \int_X \left(\int_X (r(t) - r(u))^2\,\mathrm{d}\mu_x(u)\right) \mathrm{d}\mu_x(t) \\
&\le |r|_{Lip_1}^2 \int_X \left(\int_X d^2(t,u)\mathrm{d}\mu_x(u)\right) \mathrm{d}\mu_x(t) \\
&\le |r|_{Lip_1}^2 \int_X \left(\int_X [d(t,x) + d(x,u)]^2\,\mathrm{d}\mu_x(u)\right) \mathrm{d}\mu_x(t) \\
&= |r|_{Lip_1}^2 \int_X \int_X \left\{d^2(t,x) + 2\cdot d(t,x)\cdot d(x,u) + d^2(x,u)\right\} \mathrm{d}\mu_x(u)\mathrm{d}\mu_x(t) \\
&= |r|_{Lip_1}^2 \left[\int_X d^2(t,x)\mathrm{d}\mu_x(t) + 2\int_X\int_X d(t,x)\cdot d(x,u)\mathrm{d}\mu_x(u)\mathrm{d}\mu_x(t) + \int_X d^2(x,u)\mathrm{d}\mu_x(u)\right] \\
&= |r|_{Lip_1}^2 \left[H(d^2(\cdot,x);x) + 2\left(\int_X d(t,x)\mathrm{d}\mu_x(t)\right)\cdot\left(\int_X d(u,x)\mathrm{d}\mu_x(u)\right) + H(d^2(\cdot,x);x)\right] \\
&= |r|_{Lip_1}^2 \left[H(d^2(\cdot,x);x) + 2H(d(\cdot,x);x)\cdot H(d(\cdot,x);x) + H(d^2(\cdot,x);x)\right] \\
&= |r|_{Lip_1}^2 \left[2H(d^2(\cdot,x);x) + 2H(d(\cdot,x);x)^2\right] \\
&\le |r|_{Lip_1}^2 \left[2H(d^2(\cdot,x);x) + 2H(d^2(\cdot,x);x)\right] \\
&= 4\,|r|_{Lip_1}^2 \cdot H(d^2(\cdot,x);x).
\end{aligned}$$

For r, s as above, we have the estimate

$$|T(r,s;x)| \le \sqrt{T(r,r;x)\cdot T(s,s;x)} \le 4\,|r|_{Lip_1}\cdot|s|_{Lip_1}\cdot H(d^2(\cdot,x);x).$$

Moreover, for $f \in C(X)$ and $s \in Lip_1$, the inequality

$$|T(f,s;x)| \le \sqrt{T(f,f;x)\cdot T(s,s;x)} \le 2\,\|f\|_\infty\cdot|s|_{Lip_1}\cdot\sqrt{H(d^2(\cdot,x);x)}$$

holds. Similarly, if $r \in Lip_1$ and $g \in C(X)$, we have

$$|T(r,g;x)| \le \sqrt{T(r,r;x)\cdot T(g,g;x)} \le 2\,\|g\|_\infty\cdot|r|_{Lip_1}\cdot\sqrt{H(d^2(\cdot,x);x)}.$$

Now let $f, g \in C(X)$ be fixed and $r, s \in Lip_1$ arbitrary. Then

$$\begin{aligned}
&|T(f,g;x)| \\
&= |T(f-r+r, g-s+s; x)| \\
&\leq |T(f-r, g-s; x)| + |T(f-r, s; x)| + |T(r, g-s; x)| + |T(r, s; x)| \\
&\leq \|f-r\|_\infty \cdot \|g-s\|_\infty + 2\,\|f-r\|_\infty \cdot |s|_{Lip_1} \cdot \sqrt{H(d^2(\cdot,x);x)} \\
&+ 2\,\|g-s\|_\infty \cdot |r|_{Lip_1} \cdot \sqrt{H(d^2(\cdot,x);x)} + 4\,|r|_{Lip_1} \cdot |s|_{Lip_1} \cdot H(d^2(\cdot,x);x) \\
&= \|f-r\|_\infty \cdot \left\{ \|g-s\|_\infty + 2\,|s|_{Lip_1} \cdot \sqrt{H(d^2(\cdot,x);x)} \right\} \\
&+ 2\,|r|_{Lip_1} \cdot \sqrt{H(d^2(\cdot,x);x)} \cdot \left\{ \|g-s\|_\infty + 2\,|s|_{Lip_1} \cdot \sqrt{H(d^2(\cdot,x);x)} \right\} \\
&= \left\{ \|f-r\|_\infty + 2\,|r|_{Lip_1} \sqrt{H(d^2(\cdot,x);x)} \right\} \\
&\cdot \left\{ \|g-s\|_\infty + 2\,|s|_{Lip_1} \sqrt{H(d^2(\cdot,x);x)} \right\}.
\end{aligned}$$

We now pass to the infimum over r and s, respectively, which leads us to

$$\begin{aligned}
&|T(f,g;x)| \\
&\leq K\left(\sqrt{4H(d^2(\cdot,x);x)}, f; C(X), Lip_1\right) \cdot K\left(\sqrt{4H(d^2(\cdot,x);x)}, g; C(X), Lip_1\right) \\
&= \frac{1}{2}\tilde{\omega}\left(f; 2\cdot\sqrt{4H(d^2(\cdot,x);x)}\right) \cdot \frac{1}{2}\tilde{\omega}\left(g; 2\cdot\sqrt{4H(d^2(\cdot,x);x)}\right) \\
&= \frac{1}{4}\tilde{\omega}\left(f; 4\sqrt{H(d^2(\cdot,x);x)}\right) \cdot \tilde{\omega}\left(g; 4\sqrt{H(d^2(\cdot,x);x)}\right).
\end{aligned}$$

This ends our proof. □

Similar to relation (2.1.3), we can also give a pre-Chebyshev-Grüss inequality in a compact metric space. We then apply it to the special case of the CBS operator based on $n+1$ equidistant points.

Theorem 2.2.37. *Let $X = [0,1]$ be a compact metric space, endowed with the metric $d(s,t) := |s-t|$, for $s,t \in X$. For two given functions $f, g \in C(X)$ and $x, y \in X$ fixed, the inequality*

$$\begin{aligned}
&|T(f,g;x)| \\
&\leq \frac{1}{2}\min\{\|f\|_\infty \,\widetilde{\omega_d}\,(g; 4H(|e_1-x|\,;x))\,;\|g\|_\infty\,\widetilde{\omega_d}\,(f; 4H(|e_1-x|\,;x))\}
\end{aligned}$$

holds.

Because this inequality will be applied only to the CBS operator, we will give a proof for it there.

2.2.5 Applications for (positive) linear operators

2.2.5.1 Shepard-type operators

Let $H := S_n^\mu$ in Theorem 2.2.36. Then we have the following main result, previously published in [100] (see Theorem 3.4. there):

Theorem 2.2.38. *Let $f, g \in C(X)$ be two given functions. Then the inequality*

$$|T(f,g;x)| \leq \frac{1}{4}\widetilde{\omega_d}\left(f; 4\sqrt{\sum_{i=1}^{n} \frac{d(x,x_i)^{2-\mu}}{\sum_{l=1}^{n} d(x,x_l)^{-\mu}}}\right) \cdot \widetilde{\omega_d}\left(g; 4\sqrt{\sum_{i=1}^{n} \frac{d(x,x_i)^{2-\mu}}{\sum_{l=1}^{n} d(x,x_l)^{-\mu}}}\right)$$

holds, for $x \notin \{x_1, \ldots, x_n\}$. For $x = x_i$, $|T(f,g;x)| = 0$.

Proof. If we substitute the CBS operator S_n^μ in the result of Theorem 2.2.36, the following inequality

$$\begin{aligned}|T(f,g;x)| &= \left|S_n^\mu(f \cdot g;x) - S_n^\mu(f;x) \cdot S_n^\mu(g;x)\right| \\ &\leq \frac{1}{4}\widetilde{\omega_d}\left(f; 4\sqrt{S_n^\mu(d^2(\cdot,x);x)}\right) \cdot \widetilde{\omega_d}\left(g; 4\sqrt{S_n^\mu(d^2(\cdot,x);x)}\right)\end{aligned}$$

holds. The second moment of the CBS-operator was given by equation (1.2.4). We then get the claimed result and this ends our proof. □

Remark 2.2.39. It is also possible to apply the Chebyshev-Grüss inequality for the CBS operator defined on $X = [a,b]$.

We now try to find a pre-Chebyshev-Grüss inequality for the CBS operator. For this we consider the second special case of the CBS operator, just like it was given in (1.2.5).

Taking $H := S_{n+1}^\mu$ in Theorem 2.2.37 to be the CBS operator based on $n+1$ equidistant points $x_i = \frac{i}{n}$, for $0 \leq i \leq n$ and $1 \leq \mu \leq 2$, we get:

Theorem 2.2.40. *(see Theorem 6.1. in [100]) Let $f, g \in C[0,1]$. Then the inequality*

$$\begin{aligned}&|T(f,g;x)| \\ &\leq \frac{1}{2}\min\{\|f\|_\infty \widetilde{\omega_d}\left(g; 4S_{n+1}^\mu(|e_1 - x|;x)\right); \|g\|_\infty \widetilde{\omega_d}\left(f; 4S_{n+1}^\mu(|e_1 - x|;x)\right)\}\end{aligned}$$

holds.

Proof. We want to estimate

$$|T(f,g;x)| = \left|S_{n+1}^\mu(f \cdot g;x) - S_{n+1}^\mu(f;x) \cdot S_{n+1}^\mu(g;x)\right|.$$

For two fixed functions $f, g \in C[0,1]$ and an arbitrary $s \in C^1[0,1]$, we have

$$|T(f,g;x)| = |T(f,g-s+s;x)| \leq |T(f,g-s;x)| + |T(f,s;x)|. \tag{2.2.8}$$

First, if we have $f \in C[0,1]$ and $s \in C^1[0,1]$, we continue with

$$\begin{aligned}
|T(f,s;x)| &= \left|S^{\mu}_{n+1}(f \cdot s;x) - S^{\mu}_{n+1}(f;x) \cdot S^{\mu}_{n+1}(s;x)\right| \\
&= \left|S^{\mu}_{n+1}(f(s - S^{\mu}_{n+1}(s;x));x)\right| \\
&= \left|S^{\mu}_{n+1,t}(f(t)(s(t) - s(x) + s(x) - S^{\mu}_{n+1}(s;x));x)\right| \\
&\leq \|f\|_{\infty} \cdot S^{\mu}_{n+1,t}(|s(t) - s(x)| + \left|s(x) - S^{\mu}_{n+1}(s;x)\right|;x) \\
&\leq \|f\|_{\infty} \cdot S^{\mu}_{n+1}(\left\|s'\right\|_{\infty} \cdot |e_1 - x| + \left\|s'\right\|_{\infty} \cdot S^{\mu}_{n+1}(|e_1 - x|;x);x) \\
&= 2 \cdot \|f\|_{\infty} \cdot \left\|s'\right\|_{\infty} \cdot S^{\mu}_{n+1}(|e_1 - x|;x).
\end{aligned}$$

If we now use this result in (2.2.8), we get

$$\begin{aligned}
|T(f,g;x)| &\leq \|f\|_{\infty} \cdot \|g - s\|_{\infty} + 2 \cdot \|f\|_{\infty} \cdot \left\|s'\right\|_{\infty} \cdot S^{\mu}_{n+1}(|e_1 - x|;x) \\
&= \|f\|_{\infty} \left\{\|g - s\|_{\infty} + 2 \cdot \left\|s'\right\|_{\infty} \cdot S^{\mu}_{n+1}(|e_1 - x|;x)\right\}.
\end{aligned}$$

Passing to the infimum over $s \in C^1[0,1]$, it follows

$$\begin{aligned}
|T(f,g;x)| &\leq \|f\|_{\infty} \cdot K(2 \cdot S^{\mu}_{n+1}(|e_1 - x|;x), g; C[0,1], C^1[0,1]) \\
&= \frac{1}{2} \cdot \|f\|_{\infty} \cdot \widetilde{\omega_d}\left(g, 4 \cdot S^{\mu}_{n+1}(|e_1 - x|;x)\right).
\end{aligned}$$

The same estimate holds if we interchange f and g. Putting both inequalities together, we get the result we were looking for. $\square$

In the above result, the first absolute moment of the CBS operator appears, which can be represented by

$$S^{\mu}_{n+1}(|e_1 - x|;x) = \begin{cases} \sum_{i=0}^{n} \frac{\left|x - \frac{i}{n}\right|^{1-\mu}}{\sum_{l=0}^{n}\left|x - \frac{l}{n}\right|^{-\mu}} & , x \notin \{x_0, \dots, x_n\} \\ 0 & , \text{otherwise.} \end{cases}$$

The idea is to further estimate this quantity. For that, we use an idea from [44] (see proof of Theorem 4.3).

We distinguish three important cases for different values of μ.
The first case is $\mu = 1$. The first absolute moment of the CBS operator becomes

$$\begin{aligned}
S^{1}_{n+1}(|e_1 - x|;x) &= \begin{cases} \sum_{i=0}^{n} \frac{1}{\sum_{l=0}^{n}\left|x - \frac{l}{n}\right|^{-1}} & , x \notin \{x_0, \dots, x_n\} \\ 0 & , \text{otherwise} \end{cases} \\
&= \begin{cases} (n+1)\left(\sum_{l=0}^{n} \frac{1}{\left|x - \frac{l}{n}\right|}\right)^{-1} & , x \notin \{x_0, \dots, x_n\} \\ 0 & , \text{otherwise.} \end{cases}
\end{aligned}$$

Let now l_0 be defined by $\frac{l_0}{n} < x < \frac{l_0+1}{n}$. Then we have

$$\begin{aligned}
\frac{1}{n+1}\cdot\left(\sum_{l=0}^{n}\frac{1}{\left|x-\frac{l}{n}\right|}\right) &\geq \frac{n}{n+1}\cdot\left\{\sum_{l=0}^{l_0}\frac{1}{l_0+1-l}+\sum_{l=l_0+1}^{n}\frac{1}{l-l_0}\right\}\\
&\geq \frac{n}{n+1}\left\{\int_1^{l_0+2}\frac{1}{x}\mathrm{d}x+\int_1^{n-l_0+1}\frac{1}{x}\mathrm{d}x\right\}\\
&= \frac{n}{n+1}\ln((l_0+2)\cdot(n-l_0+1))\\
&\geq \frac{n}{n+1}\cdot\ln(2n+2),
\end{aligned}$$

and the first absolute moment is then

$$S_{n+1}^{1}(|e_1-x|\,;x)\leq\frac{n+1}{n\cdot\ln(2n+2)},$$

for $x\notin\{x_0,\ldots,x_n\}$. In the end we get

$$\begin{aligned}
&|T(f,g;x)|\\
&\leq\frac{1}{2}\min\left\{\|f\|_\infty\cdot\widetilde{\omega_d}\left(g;\frac{4(n+1)}{n\cdot\ln(2n+2)}\right),\|g\|_\infty\cdot\widetilde{\omega_d}\left(f;\frac{4(n+1)}{n\cdot\ln(2n+2)}\right)\right\}.
\end{aligned}$$

For the other two cases we will consider, first let l_0 be defined by

$$\left|x-\frac{l_0}{n}\right|=\min\left\{\left|x-\frac{l}{n}\right|:0\leq l\leq n\right\}.$$

Then for the case $x\notin\{x_0,\ldots,x_n\}$, we have

$$\begin{aligned}
S_{n+1}^{\mu}(|e_1-x|\,;x) &\leq |x-x_{l_0}|^{\mu}\cdot\sum_{i=0}^{n}|x-x_i|^{1-\mu}\\
&\leq\frac{1}{n}+\frac{1}{n}\cdot\left\{\sum_{i<l_0}|x-x_i|^{1-\mu}+\sum_{i>l_0}|x-x_i|^{1-\mu}\right\}\\
&\leq\frac{1}{n}+\frac{1}{n}\cdot\left\{\sum_{k=0}^{l_0-1}\left(\frac{1}{2}+k\right)^{1-\mu}+\sum_{k=0}^{n-l_0-1}\left(\frac{1}{2}+k\right)^{1-\mu}\right\},
\end{aligned}$$

with $0\leq l_0\leq n$. Either of the two last sums may be empty. Estimating the result in the accolades from above, we get

$$S_{n+1}^{\mu}(|e_1-x|\,;x)\leq\begin{cases}\frac{1}{n}+\frac{1}{n}\cdot\left[2^{\mu}+\frac{2}{2-\mu}\cdot\left(\frac{n+1}{2}\right)^{2-\mu}\right] & \text{, for } 1<\mu<2\\ \frac{1}{n}+\frac{1}{n}\cdot[4+2\cdot\ln(n+1)] & \text{, for } \mu=2\end{cases}. \tag{2.2.9}$$

For $1<\mu<2$, we obtain

$$\begin{aligned}
&|T(f,g;x)|\\
&\leq\frac{1}{2}\min\left\{\|f\|_\infty\cdot\widetilde{\omega_d}\left(g;4S_{n+1}^{\mu}\left(|e_1-x|\,;x\right)\right),\|g\|_\infty\cdot\widetilde{\omega_d}\left(f;4S_{n+1}^{\mu}\left(|e_1-x|\,;x\right)\right)\right\}
\end{aligned}$$

where the first absolute moment can be estimated from above as in (2.2.9). For $\mu = 2$ we obtain

$$
\begin{aligned}
&|T(f,g;x)| \\
&\leq \frac{1}{2}\min\left\{\|f\|_\infty \cdot \widetilde{\omega_d}\left(g; \frac{20+8\cdot\ln(n+1)}{n}\right), \|g\|_\infty \cdot \widetilde{\omega_d}\left(f; \frac{20+8\cdot\ln(n+1)}{n}\right)\right\}.
\end{aligned}
$$

These cases and the inequalities that arise were already discussed in [100].

One can also obtain results for $\mu > 2$. This was done by G. Somorjai [106] (see also J. Szabados [111] for $\mu > 4$). If we take in [106] the positive linear operators defined there $L_n := S^{\mu}_{n+1}$, $x_i := \frac{i}{n}$ and $f \in C[0,1]$ such that $f(t) = |t-x|$, then the we need to estimate the first absolute moment $L_n(|e_1 - x|\,;x) := S^{\mu}_{n+1}(|e_1-x|\,;x)$, for $\alpha := \mu > 0$ a real number in the cited article, $0 \leq x \leq 1$, $n = 1, 2, \ldots$. What we want is to estimate

$$
S^{\mu}_{n+1}(|e_1 - x|\,;x) = \begin{cases} \sum_{i=0}^{n} \frac{\left|x-\frac{i}{n}\right|^{1-\mu}}{\sum_{l=0;l\neq i}^{n}\left|x-\frac{l}{n}\right|^{-\mu}} & \text{, for } x \notin \{x_0,\ldots,x_n\} \\ 0 & \text{, otherwise,} \end{cases}
$$

for $\mu > 2$. The method used here is similar to the one in [44]. Let l_0 be defined by

$$
|x - x_{l_0}| = \min\{|x-x_l| : 0 \leq l \leq n\}.
$$

Then for $x \notin \{x_0,\ldots,x_n\}$,

$$
\sum_{i=0}^{n} |x-x_i|^{1-\mu} \cdot \left(\frac{1}{\sum\limits_{l=0;l\neq i}^{n} |x-x_i|^{-\mu}}\right) \leq |x-x_{l_0}|^{\mu} \cdot \sum_{i=0}^{n} |x-x_i|^{1-\mu}
$$

holds, because we have

$$
\frac{1}{\sum\limits_{l=0}^{n} |x-x_l|^{-\mu}} \leq \frac{1}{|x-x_{l_0}|^{-\mu}} = |x-x_{l_0}|^{\mu}.
$$

For $i = l_0$ we get

$$
|x-x_{l_0}|^{\mu} \cdot |x-x_{l_0}|^{1-\mu} = |x-x_{l_0}| \leq \frac{1}{n},
$$

so $|x-x_{l_0}|^{\mu} \leq \left(\frac{1}{n}\right)^{\mu}$. From this we obtain

$$
\begin{aligned}
\sum_{i=0}^{n} \frac{|x-x_i|^{1-\mu}}{\sum\limits_{l=0;l\neq i}^{n} |x-x_l|^{-\mu}} &\leq \frac{1}{n} + \left(\frac{1}{n}\right)^{\mu} \cdot \left\{\sum_{i<l_0} |x-x_i|^{1-\mu} + \sum_{i>l_0} |x-x_i|^{1-\mu}\right\} \\
&= \frac{1}{n} + \left(\frac{1}{n}\right)^{\mu} \cdot \left\{\underbrace{\sum_{i=0}^{l_0-1} |x-x_i|^{1-\mu}}_{(1)} + \underbrace{\sum_{i=l_0+1}^{n} |x-x_i|^{1-\mu}}_{(2)}\right\}.
\end{aligned}
$$

For the sum (2) we have

$$\begin{aligned}
\sum_{i=l_0+1}^{n} |x-x_i|^{1-\mu} &\leq \left(\frac{1}{n}\right)^{1-\mu} \cdot \sum_{k=0}^{n-l_0-1} \left(k+\frac{1}{2}\right)^{1-\mu} \\
&= \left(\frac{1}{n}\right)^{1-\mu} \sum_{k=0}^{n-l_0-1} \left(\frac{1}{k+\frac{1}{2}}\right)^{\mu-1} \\
&\leq \left(\frac{1}{n}\right)^{1-\mu} \underbrace{\sum_{k=0}^{\infty} \left(\frac{1}{k+\frac{1}{2}}\right)^{\mu-1}}_{<c_\mu<\infty} \\
&\leq (c_\mu+1) \cdot \left(\frac{1}{n}\right)^{1-\mu},
\end{aligned}$$

because

$$\sum_{k=0}^{\infty} \left(\frac{1}{k+\frac{1}{2}}\right)^{\mu-1} = \sum_{k=0}^{\infty} \left(\frac{1}{k+1}\right)^{\mu-1} = \sum_{k=0}^{\infty} \left(\frac{1}{k}\right)^{\mu-1} = c_\mu.$$

The first sum (1) can be written as follows:

$$\begin{aligned}
\sum_{i=0}^{l_0-1} |x-x_i|^{1-\mu} &\leq \sum_{k=1}^{l_0} \left(\frac{k}{n}+\frac{1}{2n}\right)^{1-\mu} \\
&\leq \left(\frac{1}{n}\right)^{1-\mu} \cdot \sum_{k=1}^{l_0} \left(k+\frac{1}{2}\right)^{1-\mu} \\
&\leq \left(\frac{1}{n}\right)^{1-\mu} \sum_{k=0}^{l_0-1} \left(k+\frac{1}{2}\right)^{1-\mu}.
\end{aligned}$$

From both sums (1) and (2) we get

$$\begin{aligned}
&\frac{1}{n}+\left(\frac{1}{n}\right)^{\mu} \cdot \left(\frac{1}{n}\right)^{1-\mu} \cdot \left\{\sum_{k=0}^{l_0-1} \left(\frac{1}{2}+k\right)^{1-\mu} + \sum_{k=0}^{n-l_0-1} \left(\frac{1}{2}+k\right)^{1-\mu}\right\} \\
&= \frac{1}{n}+\frac{1}{n}\left\{\sum_{k=0}^{l_0-1} \left(\frac{1}{2}+k\right)^{1-\mu} + \sum_{k=0}^{n-l_0-1} \left(\frac{1}{2}+k\right)^{1-\mu}\right\} \\
&\leq \frac{1}{n}+\frac{1}{n}\left\{\sum_{k=0}^{\infty} \left(\frac{1}{2}+k\right)^{1-\mu} + \sum_{k=0}^{\infty} \left(\frac{1}{2}+k\right)^{1-\mu}\right\}.
\end{aligned}$$

A first result is

$$\sum_{i=0}^{n} \frac{|x-x_i|^{1-\mu}}{\sum_{l=0;l\neq i}^{n} |x-x_i|^{-\mu}} \leq \frac{1}{n}+\frac{2}{n}\cdot c_\mu = \frac{1}{n}\left(1+2c_\mu\right).$$

We want to find c_μ, for $\mu > 2$. For this we use the Riemann Zeta function.
We know that

$$\frac{1}{1^{\mu-1}}+\frac{1}{2^{\mu-1}}+\ldots = c_\mu = \zeta(\mu-1).$$

It is also well known, in particular, that $\zeta(2) = \sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6} \simeq 1,645$, $\zeta(3) \approx 1,202$ which is Apéry's constant, and $\zeta(4) = \frac{\pi^4}{90}$. There is also a formula due to Euler to compute $\zeta(2k)$, for $k = 1, 2, \ldots$, using Bernoulli numbers. However, no formula for $\zeta(2k+1)$ is known until now. R. Apéry has a result that states that $\zeta(3)$ is irrational, but otherwise nothing else about $\zeta(2k+1)$ is known, $k > 1$.

If in our case we give different values to $\mu - 1$, we observe that the series represented by c_μ decreases and tends to 1.

This means that

$$S_{n+1}^{\mu}(|e_1 - x|\,; x) = \sum_{i=0}^{n} \frac{|x - x_i|^{1-\mu}}{\sum_{l=0;l\neq i}^{n} |x - x_i|^{-\mu}} \leq \frac{3}{n},$$

for $x \notin \{x_0, \ldots, x_n\}$ and $\mu > 2$.

Then the pre-Chebyshev-Grüss inequality for $\mu > 2$ becomes

$$\begin{aligned} &|T(f,g;x)| \\ &\leq \frac{1}{2} \min\left\{ \|f\|_\infty \cdot \widetilde{\omega_d}\left(g; \frac{12}{n}\right); \|g\|_\infty \cdot \widetilde{\omega_d}\left(f; \frac{12}{n}\right) \right\} \end{aligned}$$

2.2.5.2 Lagrange operator

A Chebyshev-Grüss inequality for the Lagrange operator at Chebyshev nodes, similar to the one in Theorem 2.2.36, is given by:

Theorem 2.2.41. *For $f, g \in C[-1,1]$ and all $x \in [-1,1]$, the inequality*

$$\begin{aligned} |T(f,g;x)| &\leq \frac{1}{4} \|L_n\| \,(1 + \|L_n\|) \widetilde{\omega}\,(f;2) \cdot \widetilde{\omega}\,(g;2) \\ &\leq \frac{1}{2} \left(1 + \frac{3}{\pi} \log n + \frac{2}{\pi} \log^2 n\right) \omega(f;2) \cdot \omega(g;2) \end{aligned}$$

holds; here ω denotes the first order modulus.

Proof. The idea of this proof is similar to the one of Theorem 2 in [2] and that of Theorem 3.1. in [100]. Recall, however, that we have to work without the assumption of positivity. We consider the bilinear functional

$$T(f,g;x) := L_n(f \cdot g; x) - L_n(f;x) \cdot L_n(g;x).$$

Let $f, g \in C[-1,1]$ and $r, s \in Lip_1$, where $Lip_1 = \left\{ f \in C[-1,1] : \sup_{x \neq x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} < \infty \right\}$ and the seminorm on Lip_1 is defined by $|f|_{Lip_1} := \sup_{x \neq x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|}$. We are interested in estimating

$$\begin{aligned} |T(f,g;x)| &= |T(f - r + r, g - s + s; x)| \\ &\leq |T(f - r, g - s; x)| + |T(f - r, s; x)| + |T(r, g - s; x)| + |T(r, s; x)|. \end{aligned} \tag{2.2.10}$$

First note that for $f, g \in C[-1,1]$ one has

$$|T(f,g;x)| \leq \|L_n\| \,(1 + \|L_n\|)\, \|f\| \cdot \|g\|.$$

For $r, s \in Lip_1$ we have the estimate

$$\begin{aligned}
|T(r,s;x)| &= |T((r-r(0)),(s-s(0);x)| \\
&= |L_n((r-r(0))\cdot(s-s(0));x) - L_n(r-r(0);x)\cdot L_n(s-s(0);x)| \\
&\le \|L_n\|\cdot\|r-r(0)\|\cdot\|s-s(0)\| + \|L_n\|^2\cdot\|r-r(0)\|\cdot\|s-s(0)\| \\
&\le \|L_n\|\,(1+\|L_n\|)\cdot|r|_{Lip_1}\cdot|s|_{Lip_1}.
\end{aligned}$$

Moreover, for $r \in Lip_1$ and $g \in C[-1,1]$ the inequality

$$\begin{aligned}
|T(r,g;x)| &= |T(r-r(0),g;x)| \\
&= |L_n((r-r(0))\cdot g;x) - L_n(r-r(0);x)\cdot L_n(g;x)| \\
&\le \|L_n\|\cdot\|(r-r(0))\cdot g\| + \|L_n\|^2\cdot\|r-r(0)\|\cdot\|g\| \\
&\le \|L_n\|\,(1+\|L_n\|)\cdot\|g\|\cdot\|r-r(0)\| \\
&\le \|L_n\|\,(1+\|L_n\|)\cdot\|g\|\cdot|r|_{Lip_1}
\end{aligned}$$

holds. Note that in both cases considered so far we used

$$\begin{aligned}
|r(x)-r(0)| &= \frac{|r(x)-r(0)|}{|x-0|}\cdot|x-0| \\
&\le |r|_{Lip_1}\cdot|x|,
\end{aligned}$$

for $x \in [-1,1]$, i.e.,

$$\|r-r(0)\| \le |r|_{Lip_1}.$$

Similarly, if $f \in C[-1,1]$ and $s \in Lip_1$ we have

$$|T(f,s;x)| \le \|L_n\|\,(1+\|L_n\|)\cdot\|f\|\cdot|s|_{Lip_1}.$$

Then inequality (2.2.10) becomes

$$\begin{aligned}
|T(f,g;x)| &\le |T(f-r,g-s;x)| + |T(f-r,s;x)| + |T(r,g-s;x)| + |T(r,s;x)| \\
&\le \|L_n\|\,(1+\|L_n\|)\cdot\left\{\|f-r\| + |r|_{Lip_1}\right\}\cdot\{\|g-s\| + |s|_{Lip_1}\}.
\end{aligned}$$

The latter expression involves terms figuring in the K - functional

$$\begin{aligned}
&K(f,t;C[-1,1],Lip_1) \\
&= \inf\{\|f-g\| + t\cdot|g|_{Lip_1} : g \in Lip_1\},
\end{aligned}$$

for $f \in C[-1,1]$, $t \ge 0$. It is known that (see, e.g., [91])

$$K\left(f,\frac{t}{2}\right) = \frac{1}{2}\cdot\widetilde{\omega}(f;t),$$

an equality to be used in the next step.

We now pass to the infimum over r and s, respectively, which leads to

$$\begin{aligned}
|T(f,g;x)| &\le \|L_n\|\,(1+\|L_n\|)\cdot K(f,1;C,Lip_1)\cdot K(g,1;C,Lip_1) \\
&= \|L_n\|\,(1+\|L_n\|)\cdot\frac{1}{2}\cdot\widetilde{\omega}(f;2)\cdot\frac{1}{2}\cdot\widetilde{\omega}(g;2) \\
&= \frac{1}{4}\|L_n\|\,(1+\|L_n\|)\omega(f;2)\cdot\omega(g;2).
\end{aligned}$$

T. Rivlin (see [97]) proved the following inequality in the case of Lagrange interpolation at Chebyshev nodes:

$$0.9625 < \|L_n\| - \frac{2}{\pi}\log n < 1,$$

so using this result we get

$$\begin{aligned}
&\|L_n\| < \frac{2}{\pi}\log n + 1\\
&\Rightarrow 1 + \|L_n\| < 2\left(\frac{1}{\pi}\log n + 1\right)\\
&\Rightarrow \|L_n\|\left(1 + \|L_n\|\right) < 2\left(1 + \frac{3}{\pi}\log n + \frac{2}{\pi^2}\log^2 n\right)
\end{aligned}$$

which implies the result. □

Remark 2.2.42. If $L_n(f\cdot g;x) = (f\cdot g)(x) = f(x)\cdot g(x)$ and $L_n(f;x) = f(x)$, $L_n(g;x) = g(x)$, the left hand side of the inequality vanishes, while the right hand side grows logarithmically. The above theorem, together with the proof, can be found in [58], paper that was submitted for publication.

2.2.6 Chebyshev-Grüss inequalities via discrete oscillations

A Chebyshev-Grüss inequality via discrete oscillations is introduced, one that in some sense improves the classical inequalities known until now in the literature and gives a different approach. On the other hand, there are cases when Chebyshev-Grüss inequalities, involving the least concave majorant of the modulus of continuity (see for ex. Theorem 2.2.3), give better estimates. In our article [58], we presented this approach and gave the applications that follow.

2.2.6.1 The compact topological space case

Let μ be a (not necessarily positive) Borel measure on the compact topological space X.

Let $\int\limits_X \mathrm{d}\mu(x) = 1$, and set $L(f) = \int\limits_X f(x)\mathrm{d}\mu(x)$, for $f \in C(X)$. Then, for $f, g \in C(X)$, we have

$$\begin{aligned}
L(fg) - L(f)L(g) &= \int\limits_X f(x)g(x)\mathrm{d}\mu(x) - \int\limits_X f(x)\mathrm{d}\mu(x)\cdot\int\limits_X g(y)\mathrm{d}\mu(y)\\
&= \iint\limits_{X^2} f(x)g(x)\mathrm{d}(\mu\otimes\mu)(x,y) - \iint\limits_{X^2} f(x)g(y)\mathrm{d}(\mu\otimes\mu)(x,y)\\
&= \iint\limits_{X^2} f(x)(g(x) - g(y))\mathrm{d}(\mu\otimes\mu)(x,y).
\end{aligned}$$

Similarly,

$$L(fg) - L(f)L(g) = \iint\limits_{X^2} f(y)(g(y) - g(x))\mathrm{d}(\mu\otimes\mu)(x,y).$$

By addition,

$$2(L(fg) - L(f)L(g)) = \iint\limits_{X^2} (f(x) - f(y))(g(x) - g(y)) \mathrm{d}(\mu \otimes \mu)(x, y). \qquad (2.2.11)$$

Let

$$osc_L(f) := \max\{|f(x) - f(y)| : (x, y) \in supp(\mu \otimes \mu)\},$$

where $supp(\mu \otimes \mu)$ is the support of the tensor product of the Borel measure μ with itself (see [6]), and let $\Delta := \{(x, x) : x \in X\}$. From (2.2.11) we get

$$L(fg) - L(f)L(g) = \frac{1}{2} \iint\limits_{X^2 \setminus \Delta} (f(x) - f(y))(g(x) - g(y)) \mathrm{d}(\mu \otimes \mu)(x, y).$$

Then we have the following result.

Theorem 2.2.43. *The Chebyshev-Grüss inequality in this case is given by*

$$|L(fg) - L(f)L(g)| \leq \frac{1}{2} \cdot osc_L(f) \cdot osc_L(g) \, |\mu \otimes \mu| \, (X^2 \setminus \Delta),$$

for $f, g \in C(X)$ and $|\mu \otimes \mu|$ is the absolute value of the tensor product of the Borel measure μ with itself (see Chapter 1 in [6]).

Example 2.2.44. Let $X = [0, 1]$ and consider the functional

$$L(f) = a \int_0^1 f(t) \mathrm{d}t + (1 - a) f\left(\frac{1}{2}\right), \text{ for } 0 \leq a \leq 1.$$

Then $L(f) = \int_0^1 f(t) \mathrm{d}\mu$, where the Borel measure μ is given by

$$\mu = a\lambda + (1 - a)\varepsilon_{\frac{1}{2}}$$

on X, with λ the Lebesgue measure on $[0, 1]$ and $\varepsilon_{\frac{1}{2}}$ the measure concentrated at $\frac{1}{2}$. Then the tensor product of μ with itself is

$$\begin{aligned} \mu \otimes \mu &= \left(a\lambda + (1 - a)\varepsilon_{\frac{1}{2}}\right) \otimes \left(a\lambda + (1 - a)\varepsilon_{\frac{1}{2}}\right) \\ &= a^2(\lambda \otimes \lambda) + a(1 - a)(\lambda \otimes \varepsilon_{\frac{1}{2}}) + (1 - a)a(\varepsilon_{\frac{1}{2}} \otimes \lambda) + (1 - a)^2(\varepsilon_{\frac{1}{2}} \otimes \varepsilon_{\frac{1}{2}}). \end{aligned}$$

$\mu \otimes \mu$ is a positive measure, so $|\mu \otimes \mu| = \mu \otimes \mu$, and

$$\begin{aligned} \mu \otimes \mu \left([0, 1]^2 \setminus \Delta\right) &= [a^2(\lambda \otimes \lambda) + a(1 - a)(\lambda \otimes \varepsilon_{\frac{1}{2}}) \\ &+ a(1 - a)(\varepsilon_{\frac{1}{2}} \otimes \lambda) + (1 - a)^2(\varepsilon_{\frac{1}{2}} \otimes \varepsilon_{\frac{1}{2}})] \left([0, 1]^2 \setminus \Delta\right) \\ &= a^2 + 2a(1 - a) = a(2 - a). \end{aligned}$$

The inequality becomes :

$$|L(fg) - L(f)L(g)| \leq \frac{1}{2} \cdot a(2 - a) \cdot osc_L(f) \cdot osc_L(g),$$

for two functions $f, g \in C[0, 1]$.

2.2.6.2 The discrete linear functional case

Let X be an arbitrary set and $B(X)$ the set of all real-valued, bounded functions on X. Take $a_n \in \mathbb{R}$, $n \geq 0$, such that $\sum_{n=0}^{\infty} |a_n| < \infty$ and $\sum_{n=0}^{\infty} a_n = 1$. Furthermore, let $x_n \in X$, $n \geq 0$, be arbitrary mutually distinct points of X. For $f \in B(X)$ set $f_n := f(x_n)$. Now consider the functional $L : B(X) \to \mathbb{R}$, $Lf = \sum_{n=0}^{\infty} a_n f_n$. L is linear and $Le_0 = 1$.

Then the relations

$$\begin{aligned}
L(f \cdot g) - L(f) \cdot L(g) &= \sum_{n=0}^{\infty} a_n f_n g_n - \sum_{n=0}^{\infty} a_n f_n \cdot \sum_{m=0}^{\infty} a_m g_m \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} a_m \right) a_n f_n g_n - \sum_{n,m=0}^{\infty} a_n a_m f_n g_m \\
&= \sum_{n=0}^{\infty} a_n^2 f_n g_n + \sum_{n,m=0;m\neq n}^{\infty} a_m a_n f_n g_n \\
&- \sum_{n=0}^{\infty} a_n^2 f_n g_n - \sum_{n,m=0;m\neq n}^{\infty} a_n a_m f_n g_m \\
&= \sum_{n,m=0;m\neq n}^{\infty} a_n a_m f_n (g_n - g_m) \\
&= \sum_{0\leq n<m<\infty} a_n a_m f_n (g_n - g_m) + \sum_{0\leq n>m<\infty} a_n a_m f_n (g_n - g_m) \\
&= \sum_{0\leq n<m<\infty} a_n a_m f_n (g_n - g_m) - \sum_{0\leq n<m<\infty} a_n a_m f_m (g_n - g_m) \\
&= \sum_{0\leq n<m<\infty} a_n a_m (f_n - f_m)(g_n - g_m)
\end{aligned}$$

hold.

Theorem 2.2.45. *The Chebyshev-Grüss inequality for the above linear, not necessarily positive, functional L is given by:*

$$|L(fg) - L(f) \cdot L(g)| \leq osc_L(f) \cdot osc_L(g) \cdot \sum_{0\leq n<m<\infty} |a_n a_m|,$$

where $f, g \in B(X)$ and the oscillations are given by:

$$\begin{aligned}
osc_L(f) &:= \sup\{|f_n - f_m| : 0 \leq n < m < \infty\}, \\
osc_L(g) &:= \sup\{|g_n - g_m| : 0 \leq n < m < \infty\}.
\end{aligned}$$

Theorem 2.2.46. *In particular, if $a_n \geq 0$, $n \geq 0$, then L is a positive linear functional and we have:*

$$|L(fg) - L(f) \cdot L(g)| \leq \frac{1}{2} \cdot \left(1 - \sum_{n=0}^{\infty} a_n^2 \right) \cdot osc_L(f) \cdot osc_L(g),$$

for $f, g \in B(X)$ and the oscillations given as above.

Remark 2.2.47. The above inequality is sharp in the sense that we can find a functional L such that equality holds.

Example 2.2.48. Let us consider the following functional

$$Lf := (1-a)f(0) + af(1), \text{ for } 0 \le a \le 1.$$

For this functional we have

$$\begin{aligned}
&L(fg) - Lf \cdot Lg \\
&= (1-a)f(0)g(0) + af(1)g(1) - [(1-a)f(0) + af(1)] \cdot [(1-a)g(0) + ag(1)],
\end{aligned}$$

so after some calculations we get that the left-hand side is

$$\begin{aligned}
|L(fg) - Lf \cdot Lg| &= \left| \underbrace{a(1-a)}_{\ge 0} \cdot [f(0) - f(1)] \cdot [g(0) - g(1)] \right| \\
&= a(1-a)\, |f(0) - f(1)| \cdot |g(0) - g(1)|
\end{aligned}$$

and the right-hand side is

$$\begin{aligned}
&\frac{1}{2}\left(1 - \sum_{n=0}^{\infty} a_n^2\right) \cdot osc_L(f) \cdot osc_L(g) \\
&= \frac{1}{2} \cdot [1 - a^2 - (1-a)^2] \cdot |f(0) - f(1)| \cdot |g(0) - g(1)| \\
&= a(1-a)\, |f(0) - f(1)| \cdot |g(0) - g(1)| \,.
\end{aligned}$$

2.2.7 Applications for (positive) linear operators

2.2.7.1 The Bernstein operator

Consider the classical Bernstein operators

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x),\ f \in \mathbb{R}^{[0,1]},\ x \in [0,1],$$

where $b_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$. According to Theorem 2.2.46, for each $x \in [0,1]$, $f, g \in B[0,1]$ we have

$$|B_n(f \cdot g)(x) - B_n f(x) \cdot B_n g(x)| \le \frac{1}{2}\left(1 - \sum_{k=0}^{n} b_{n,k}^2(x)\right) \cdot osc_{B_n}(f) \cdot osc_{B_n}(g), \quad (2.2.12)$$

where

$$osc_{B_n}(f) := \max\{|f_k - f_l| : 0 \le k < l \le n\}$$

and $f_k := f\left(\frac{k}{n}\right)$; similar definitions apply to g.

Example 2.2.49. If we consider $f, g \in B[0,1]$ to be Dirichlet functions defined by

$$f(x) := \begin{cases} 1 & \text{for } x \in \mathbb{Q}, \\ 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

and analogously for g, with $f_k := f\left(\frac{k}{n}\right)$ (the same for g), then we observe that the oscillations in the above inequality vanish, so the right hand-side is zero. In other words, in this case (2.2.12) is a very good estimate and covers a case which cannot be handled by the use of the least concave majorant, as the function f is not in $C[0,1]$.

We are now interested in estimating the sum of the squares of the fundamental functions of the Bernstein operator. In order to do this, let $\varphi_n(x) := \sum_{k=0}^{n} b_{n,k}^2(x)$, $x \in [0,1]$. Since

$$\left(\frac{1}{n+1}\sum_{k=0}^{n} b_{n,k}^2(x)\right)^{\frac{1}{2}} \geq \frac{1}{n+1}\sum_{k=0}^{n} b_{n,k}(x) = \frac{1}{n+1},$$

we get

$$\varphi_n(x) \geq \frac{1}{n+1},\ x \in [0,1], \tag{2.2.13}$$

and therefore

$$|B_n(f\cdot g)(x) - B_n f(x)\cdot B_n g(x)| \leq \frac{n}{2(n+1)}\cdot osc_{B_n}(f)\cdot osc_{B_n}(g),\ x \in [0,1]. \tag{2.2.14}$$

Let us remark that equality is attained in (2.2.13) if and only if $n = 1$ and $x = \frac{1}{2}$. In fact, inspired also by some computations with Maple, we state the following conjectures:

Conjecture 2.2.50. *φ_n is convex on $[0,1]$.*

Conjecture 2.2.51. *φ_n is decreasing on $\left[0,\frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2},1\right]$.*

Conjecture 2.2.52. *$\varphi_n(x) \geq \varphi_n\left(\frac{1}{2}\right)$, $x \in [0,1]$.*

Since $\varphi_n\left(\frac{1}{2}-t\right) = \varphi_n\left(\frac{1}{2}+t\right)$, $t \in \left[0,\frac{1}{2}\right]$, we see that
Conjecture 2.2.50 $\Rightarrow$ Conjecture 2.2.51 $\Rightarrow$ Conjecture 2.2.52.
On the other hand, it can be proven that

$$\varphi_n\left(\frac{1}{2}\right) = 4^{-n}\binom{2n}{n},\ \varphi_n'\left(\frac{1}{2}\right) = 0,\ \varphi_n''\left(\frac{1}{2}\right) = 4^{2-n}\binom{2n-2}{n-1} > 0,$$

and so $\frac{1}{2}$ is a minimum point for φ_n. Conjecture 2.2.52 claims that it is an absolute minimum point; in other words,

$$\varphi_n(x) \geq \frac{1}{4^n}\binom{2n}{n},\ x \in [0,1]. \tag{2.2.15}$$

The following confirmation of Conjecture 2.2.52 is due to Dr. Th. Neuschel (Katholieke Universiteit Leuven).

Lemma 2.2.53. *For $n \in \mathbb{N}$ and $x \in [0,1]$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}^2 x^{2k}(1-x)^{2(n-k)} \geq \frac{1}{4^n}\binom{2n}{n}.$$

Proof. For symmetry reasons, it suffices to prove the statement only for $0 \leq x \leq \frac{1}{2}$. In the sequel we denote P_n to be the $n-$th Legendre polynomial, given by

$$P_n(x) := \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2 (x+1)^k (x-1)^{n-k}.$$

We make a change of variable, namely set $y := \frac{1-2x+2x^2}{1-2x} \geq 1$ and we get

$$\left(y - \sqrt{y^2-1}\right)^n \cdot P_n(y) = \sum_{k=0}^{n} \binom{n}{k}^2 x^{2k}(1-x)^{2(n-k)} = \varphi_n(x),$$

so we have to show that

$$\left(y - \sqrt{y^2-1}\right)^n \cdot P_n(y) \geq \frac{1}{4^n}\binom{2n}{n}$$

holds, for $y \geq 1$. The inequality holds for $y = 1$ and $y = \infty$. In the last case, the inequality is even sharp. Now it is enough to show :

$$\frac{d}{dy}\{(y - \sqrt{y^2-1})^n P_n(y)\} \leq 0 \text{ for } y > 1.$$

This is equivalent to the following statement:

$$P_n'(y) \leq \frac{n}{\sqrt{y^2-1}} P_n(y) \text{ for } y > 1.$$

Using the formula

$$\frac{y^2-1}{n} P_n'(y) = yP_n(y) - P_{n-1}(y),$$

we now have to prove the following:

$$(y - \sqrt{y^2-1})P_n(y) \leq P_{n-1}(y) \text{ for } y > 1,$$

which is equivalent to

$$P_n(y) \leq (y + \sqrt{y^2-1})P_{n-1}(y) \text{ for } y > 1. \tag{2.2.16}$$

The inequality (2.2.16) can be proven by induction. For $n = 1$ the inequality holds. We assume that the inequality holds also for n and we want to show:

$$P_{n+1}(y) \leq (y + \sqrt{y^2-1})P_n(y) \text{ for } y > 1.$$

Using Bonnet's recursion formula

$$P_{n+1}(y) = \frac{2n+1}{n+1} yP_n(y) - \frac{n}{n+1} P_{n-1}(y),$$

we now have to show that the following holds:

$$\left(\frac{2n+1}{n+1} y - (y + \sqrt{y^2-1})\right) P_n(y) \leq \frac{n}{n+1} P_{n-1}(y).$$

After evaluation

$$\begin{aligned}\left(\frac{2n+1}{n+1} y - (y + \sqrt{y^2-1})\right) P_n(y) &\leq \frac{n}{n+1}(y - \sqrt{y^2-1})P_n(y)\\ &\leq \frac{n}{n+1}(y - \sqrt{y^2-1})(y + \sqrt{y^2-1})P_{n-1}(y)\\ &= \frac{n}{n+1} P_{n-1}(y),\end{aligned}$$

we obtain the result. □

Actually, Dr. Th. Neuschel also validated Conjecture 2.2.51 (see the paper of G. Nikolov [88] for more details). Conjecture 2.2.50 was discussed and proved in recent papers by I. Gavrea and M. Ivan in [41], and by G. Nikolov in [88], independently. Conjecture 2.2.52 is the weakest of the three, and it is also the one of interest for us.

In order to compare (2.2.13) and (2.2.15), it is not difficult to prove the inequalities

$$\frac{1}{n+1} < \frac{1}{2\sqrt{n}} < \frac{1}{4^n}\binom{2n}{n} < \frac{1}{\sqrt{2n+1}},\ n \geq 2.$$

More precise inequalities can be found in [35]:

$$\frac{1}{\sqrt{\pi(n+3)}} < \frac{1}{4^n}\binom{2n}{n} < \frac{1}{\sqrt{\pi(n-1)}},\ n \geq 2.$$

Because we have proven that Conjecture 2.2.52 is true, we have the following result.

Theorem 2.2.54. *The Chebyshev-Grüss inequality for the Bernstein operator is:*

$$|B_n(f \cdot g)(x) - B_n f(x) \cdot B_n g(x)| \leq \frac{1}{2}\left(1 - \frac{1}{4^n}\binom{2n}{n}\right) \cdot osc_{B_n}(f) \cdot osc_{B_n}(g), \quad (2.2.17)$$

for $x \in [0,1]$.

We now compare this new approach with the classical Chebyshev-Grüss inequalities for the Bernstein operator (2.2.4) and (2.2.5) from Theorem 2.2.11.

Remark 2.2.55. In (2.2.12) and (2.2.4), the right-hand sides depend on x and vanish when $x \to 0$ or $x \to 1$. Their maximum values, as functions of x, are attained for $x = \frac{1}{2}$, and (2.2.14), (2.2.17), (2.2.5) illustrate this fact. On the other hand, in (2.2.12) the oscillations of f and g are relative only to the points $0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1$, while in (2.2.4) the oscillations, expressed in terms of $\widetilde{\omega}$, are relative to the whole interval $[0,1]$. Of course, those approaches can only be compared for functions in $C[0,1]$. It can also be shown that in this case, the estimate in terms of discrete oscillations yields a result quite similar to (2.2.4).

2.2.7.2 King operators

We need $\sum_{k=0}^{n}(v_{n,k}^*(x))^2$ to be minimal. Let $\varphi_n(x) := \sum_{k=0}^{n}(v_{n,k}^*(x))^2$.

For $n = 1$, we have that

$$\begin{aligned}\varphi_1(x) &= \sum_{k=0}^{1}(v_{1,k}^*(x))^2 = (v_{1,0}^*(x))^2 + (v_{1,1}^*(x))^2 \\ &= 2x^4 - 2x^2 + 1\end{aligned}$$

and this attains its minimum for $x = \frac{\sqrt{2}}{2}$. This minimum is

$$\varphi_1\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}.$$

Theorem 2.2.56. *The Chebyshev-Grüss inequality for $n = 1$ then looks as follows:*

$$\begin{aligned}|T(f,g;x)| &\le \frac{1}{4}\cdot osc_{V_1^*}(f)\cdot osc_{V_1^*}(g)\\ &= \frac{1}{4}\cdot |f_0 - f_1|\cdot |g_0 - g_1|\,.\end{aligned}$$

For $n = 2, 3, \ldots$, the problem of finding the minimum is more difficult, since

$$\begin{aligned}\varphi_n(x) &= \sum_{k=0}^{n} (v_{n,k}^*(x))^2\\ &= \sum_{k=0}^{n} \binom{n}{k}^2 (r_n^*(x))^{2k}(1 - r_n^*(x))^{2(n-k)}.\end{aligned}$$

In any case, the estimate

$$\varphi_n(x) = \sum_{k=0}^{n} (v_{n,k}^*(x))^2 \ge \frac{1}{n+1}$$

holds, for $x \in [0,1]$ and $n = 2, 3, \ldots$. As a proof for this,

$$\sqrt{\frac{\sum_{k=0}^{n} v_{n,k}^*(x))^2}{n+1}} \ge \frac{\sum_{k=0}^{n} v_{n,k}^*(x)}{n+1} = \frac{1}{n+1}.$$

Then we get

$$1 - \sum_{k=0}^{n} (v_{n,k}^*(x))^2 \le 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Theorem 2.2.57. *For $n = 2, 3, \ldots$ there holds*

$$|V_n^*(fg)(x) - V_n^*(f;x)\cdot V_n^*(g;x)| \le \frac{n}{2(n+1)}\cdot osc_{V_n^*}(f)\cdot osc_{V_n^*}(g).$$

2.2.7.3 Piecewise linear interpolation at equidistant knots

In this case, we need to find the minimum of the sum $\tau_n := \sum_{i=0}^{n} u_{i,n}^2$. For a particular interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, we get that

$$\tau_n(x) := \sum_{i=0}^{n} u_{i,n}^2(x) = (nx - i + 1)^2 + (i - nx)^2, \text{ for } i = 1, \ldots, n.$$

For $i = 1$, $x \in \left[0, \frac{1}{n}\right]$ and $\tau_n(x) = (nx-1)^2$, while for $i = n$, $x \in \left[\frac{n-1}{n}, 1\right]$ and $\tau_n(x) = (nx - n + 1)^2$. So $\tau_n(x) = (nx - i + 1)^2 + (i - nx)^2$ is minimum if and only if $x = \frac{2i-1}{2n}$ and the minimum value of $\tau_n(x)$ is $\frac{1}{2}$.

Theorem 2.2.58. *The Chebyshev-Grüss inequality for S_{Δ_n} is*

$$\begin{aligned}|S_{\Delta_n}(f\cdot g)-S_{\Delta_n}(f)\cdot S_{\Delta_n}(g)| &\le \frac{1}{2}\left(1-\sum_{i=0}^{n}u_{n,i}^2(x)\right)\cdot osc_{S_{\Delta_n}}(f)\cdot osc_{S_{\Delta_n}}(g)\\ &\le \frac{1}{2}\left(1-\frac{1}{2}\right)\cdot osc_{S_{\Delta_n}}(f)\cdot osc_{S_{\Delta_n}}(g)\\ &= \frac{1}{4}\cdot osc_{S_{\Delta_n}}(f)\cdot osc_{S_{\Delta_n}}(g),\end{aligned}$$

with

$$\begin{aligned}osc_{S_{\Delta_n}}(f) &:= \max\left\{|f_k-f_l| : 0\le k<l\le n\right\}\\ osc_{S_{\Delta_n}}(g) &:= \max\left\{|g_k-g_l| : 0\le k<l\le n\right\},\end{aligned}$$

where $f_k := f\left(\frac{k}{n}\right)$.

Remark 2.2.59. This inequality implies a classical Chebyshev-Grüss-type inequality because $|f_k-f_l|\le M-m$ and $|g_k-g_l|\le P-p$, respectively. Here M,P denote upper bounds of f and g, and m,p lower ones. It is easy to give examples in which this approach via discrete oscillations gives strictly better inequalities.

2.2.7.4 The Mirakjan-Favard-Szász operator

This is our first application of Theorem 2.2.46 for operators defined for functions given on an infinite interval. We set

$$\sigma_n(x) := e^{-2nx}\sum_{k=0}^{\infty}\frac{(nx)^{2k}}{(k!)^2}$$

and we want to find the infimum:

$$\inf_{x\ge 0}\sigma_n(x) := \iota \ge 0.$$

Because $\sigma_n(x)\ge\iota$, we obtain the following result.

Theorem 2.2.60. *For the Mirakjan-Favard-Szász operator we have*

$$\begin{aligned}|T(f,g;x)| &\le \frac{1}{2}\left(1-\sigma_n(x)\right)\cdot osc_{M_n}(f)\cdot osc_{M_n}(g)\\ &\le \frac{1}{2}(1-\iota)\cdot osc_{M_n}(f)\cdot osc_{M_n}(g),\end{aligned}$$

where $f,g\in C_b[0,\infty)$, $osc_{M_n}(f)=\sup\{|f_k-f_l| : 0\le k<l<\infty\}$, with $f_k := f\left(\frac{k}{n}\right)$ and a similar definition applying to g. $C_b[0,\infty)$ is the set of all continuous, real-valued, bounded functions on $[0,\infty)$.

Lemma 2.2.61. *The relation*

$$\inf_{x\ge 0}\sigma_n(x) = \iota = 0.$$

holds.

Proof. We first need to prove that

$$\lim_{x\to\infty} e^{-2nx} I_0(2nx) = 0$$

holds, for a fixed n and I_0 being the modified Bessel function of the first kind of order 0 (see the paper by E. Berdysheva [12] and the references therein). The power series expansion for modified Bessel functions of the first kind of order 0 is

$$I_0(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(k!)^2},$$

so for a fixed n we have

$$I_0(2nx) = \sum_{k=0}^{\infty} \frac{(nx)^{2k}}{(k!)^2}$$

and

$$e^{-2nx} \cdot I_0(2nx) = e^{-2nx} \cdot \sum_{k=0}^{\infty} \frac{(nx)^{2k}}{(k!)^2} = \sigma_n(x).$$

We now use Lebesgue's dominated convergence theorem and the integral expression

$$\begin{aligned} I_0(2nx) &= \frac{1}{\pi}\int_{-1}^{1} e^{-2ntx} \cdot \frac{1}{\sqrt{1-t^2}}\mathrm{d}t, \\ e^{-2nx} \cdot I_0(2nx) &= \frac{1}{\pi}\int_{-1}^{1} e^{-2nx(1+t)} \cdot \frac{1}{\sqrt{1-t^2}}\mathrm{d}t, \end{aligned}$$

for n fixed and we conclude that $\sigma_n(x) \to 0$, as $x \to \infty$, because we see from above that $e^{-2nx} \cdot I_0(2nx) \to 0$, for $x \to \infty$. □

Corollary 2.2.62. *The Chebyshev-Grüss inequality for the Mirakjan-Favard-Szász operator is:*

$$|T(f,g;x)| \leq \frac{1}{2} \cdot osc_{M_n}(f) \cdot osc_{M_n}(g),$$

where $f, g \in C_b[0,\infty)$, $osc_{M_n}(f) = \sup\{|f_k - f_l| : 0 \leq k < l < \infty\}$ and a similar definition applying to g.

2.2.7.5 The Baskakov operator

We set

$$\vartheta_n(x) := \frac{1}{(1+x)^{2n}} \sum_{k=0}^{\infty} \binom{n+k-1}{k}^2 \left(\frac{x}{1+x}\right)^{2k}, \text{ for } x \geq 0.$$

We need to find the infimum:

$$\inf_{x\geq 0} \vartheta_n(x) := \epsilon \geq 0.$$

Because $\vartheta_n(x) \geq \epsilon$, we obtain the following result.

Theorem 2.2.63. *For the Baskakov operator one has*

$$\begin{aligned}|T(f,g;x)| &\le \frac{1}{2}\left(1-\vartheta_n(x)\right)\cdot osc_{A_n}(f)\cdot osc_{A_n}(g)\\ &\le \frac{1}{2}(1-\epsilon)\cdot osc_{A_n}(f)\cdot osc_{A_n}(g),\end{aligned}$$

where $f,g \in C_b[0,\infty)$, $osc_{A_n}(f) = \sup\{|f_k - f_l| : 0 \le k < l < \infty\}$, $f_k := f\left(\frac{k}{n}\right)$ and a similar definition applying to g.

Lemma 2.2.64. *The relation $\inf_{x\ge 0}\vartheta_n(x) = \epsilon = 0$ holds, for all $n \ge 1$.*

Proof. In [12] the following functions were defined. For $I_c = [0,\infty)(c \in \mathbb{R}, c \ge 0)$, $n > 0$, $k \in \mathbb{N}_0$ and $x \in I_c$, we have

$$p_{n,k}^{[c]}(x) := (-1)^k \binom{-\frac{n}{c}}{k}(cx)^k(1+cx)^{-\frac{n}{c}-k},\ c \ne 0.$$

For $c = 1$, we get

$$\begin{aligned}p_{n,k}^{[1]}(x) = p_{n,k}(x) &= (-1)^k\binom{-n}{k}x^k(1+x)^{-n-k}\\ &= \binom{n+k-1}{k}x^k(1+x)^{-n-k} =: a_{n,k}(x),\end{aligned}$$

so we obtain the fundamental functions of the Baskakov operator. The following kernel function was defined in [12]

$$T_{n,c}(x,y) = \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x)\cdot p_{n,k}^{[c]}(y),\ \text{for } x,y \in I_c.$$

We are interested in the case $c = 1$ and $x = y$, so the above kernel becomes

$$T_{n,1}(x,x) = \sum_{k=0}^{\infty} p_{n,k}^2(x) = \sum_{k=0}^{\infty} a_{n,k}^2(x) =: \vartheta_n(x).$$

For $n = 1$, we get

$$\vartheta_1(x) = T_{1,1}(x,x) = \frac{1}{(1+x)^2}\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^{2k} = \frac{1}{1+2x} \longrightarrow 0,\ \text{for } x \to \infty.$$

For $n > 1$,

$$T_{n,1}(x,x) = \frac{1}{\pi}\int_0^1 (\phi(x,x,t))^n \frac{\mathrm{d}t}{\sqrt{t(1-t)}},$$

where, for $\phi(x,x,t) = [1+4x(1-t)+4x^2(1-t)]^{-1}$, it holds:

$$0 < \phi(x,x,t) \le 1, \forall t \in [0,1], \forall x \ge 0.$$

Therefore

$$T_{2,1}(x,x) \ge T_{3,1}(x,x) \ge T_{4,1}(x,x) \ge \ldots \ge 0, \forall x \ge 0. \tag{2.2.18}$$

Now for $n = 2$, we have

$$T_{2,1}(x,x) = \sum_{k=0}^{\infty} p_{2,k}^2(x) = \frac{1}{(1+x)^4} \sum_{k=0}^{\infty} (k+1)^2 \left(\frac{x}{1+x}\right)^{2k}.$$

Let $\left(\frac{x}{1+x}\right)^2 = y$. Then

$$\sum_{k=0}^{\infty} (k+1)^2 y^k = \frac{1+y}{(1-y)^3}.$$

Thus

$$T_{2,1}(x,x) = \frac{2x^2+2x+1}{(2x+1)^3} \to 0, \text{ for } x \to \infty. \tag{2.2.19}$$

For $n \geq 3$ it follows from (2.2.18) that $0 \leq T_{n,1}(x,x) \leq T_{2,1}(x,x)$. Combining this with (2.2.19), we get

$$\lim_{x \to 0} T_{n,1}(x,x) = 0, \forall n \geq 1,$$

and so the proof is finished. □

An inequality analogous to the one in Corollary 2.2.62 is now immediate.

2.2.7.6 The Bleimann-Butzer-Hahn operators

We set

$$\psi_n(t) = \frac{1}{(1+t)^{2n}} \sum_{k=0}^{n} \binom{n}{k}^2 t^{2k},$$

for $t \geq 0$. We make a change of variable, namely set $x = \frac{t}{t+1} \in [0,1)$. Then we get

$$\begin{aligned}\psi_n(t) &= \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{t}{t+1}\right)^{2k} \left(\frac{1}{t+1}\right)^{2n-2k} \\ &= \sum_{k=0}^{n} \binom{n}{k}^2 x^{2k}(1-x)^{2n-2k}.\end{aligned}$$

So $\psi_n(t) = \varphi_n(x)$, i.e., $\inf_{t \geq 0} \psi_n(t) = \inf_{x \in [0,1]} \varphi_n(x) = \frac{1}{4^n}\binom{2n}{n}$, as shown in Lemma 2.2.53.

This leads to

Theorem 2.2.65. *The Chebyshev-Grüss inequality in this case is:*

$$|T(f,g;x)| \leq \frac{1}{2}\left(1 - \frac{1}{4^n}\binom{2n}{n}\right) \cdot osc_{BH_n}(f) \cdot osc_{BH_n}(g),$$

with $f, g \in C_b[0,\infty)$, $x \in [0,\infty)$ and

$$osc_{BH_n}(f) := \sup\left\{|f_k - f_l| : 0 \leq k < l \leq n\right\},$$

for $f_k := f\left(\frac{k}{n-k+1}\right)$ and a similar definition applying to g.

2.2.7.7 The Lagrange operator

If we consider the Langrange interpolation operator formed upon any infinite matrix X and relation (1.3.1), then we get the following result:

Theorem 2.2.66. *The Chebyshev-Grüss inequality with discrete oscillations for the Lagrange operator is given by*

$$|T(f,g;x)| \leq osc_{L_n}(f) \cdot osc_{L_n}(g) \cdot \left(\frac{\Lambda_n^2(x)}{2} - \frac{1}{8} \right)$$

for $f, g \in B[-1,1]$ and $-1 \leq x \leq 1$.

Proof. We need to estimate the sum

$$\begin{aligned} \sum_{1 \leq k < m \leq n} |l_{k,n}(x) \cdot l_{m,n}(x)| &= \left[\left(\sum_{i=1}^{n} |l_{i,n}(x)| \right)^2 - \left(\sum_{i=1}^{n} l_{i,n}^2(x) \right) \right] /2 \\ &= \left[\Lambda_n^2(x) - \left(\sum_{i=1}^{n} l_{i,n}^2(x) \right) \right] /2. \end{aligned}$$

From (1.3.1), we know that

$$\sum_{i=1}^{n} l_{i,n}^2(x) \geq \frac{1}{4},$$

so we get

$$\sum_{1 \leq k < m \leq n} |l_{k,n}(x) \cdot l_{m,n}(x)| \leq \frac{\Lambda_n^2(x)}{2} - \frac{1}{8},$$

and this ends our proof. □

2.2.7.8 The Lagrange operator at Chebyshev nodes

If we consider Chebyshev nodes, we have the following theorem:

Theorem 2.2.67. *For $f, g \in B[-1,1]$ and $x \in [-1,1]$ fixed, the following inequality*

$$\begin{aligned} |T(f,g;x)| &\leq osc_{L_n}(f) \cdot osc_{L_n}(g) \cdot \sum_{1 \leq k < m \leq n} |l_{k,n}(x) \cdot l_{m,n}(x)| \\ &\leq osc_{L_n}(f) \cdot osc_{L_n}(g) \cdot \left\{ \frac{\Lambda_n^2(x) - c \left[1 + (\cos^2 nt) \cdot \frac{\pi^2}{6} \right]}{2} \right\} \\ &\leq osc_{L_n}(f) \cdot osc_{L_n}(g) \cdot \left\{ \frac{\Lambda_n^2(X) - c \left[1 + (\cos^2 nt) \cdot \frac{\pi^2}{6} \right]}{2} \right\} \end{aligned}$$

holds, for a suitable constant c and $x = \cos t$. Here we recall the asymptotic result given for the Lebesgue constant

$$\Lambda_n^2(X) := \left[\frac{2}{\pi} \log n + \frac{2}{\pi} \left(\log \frac{8}{\pi} + \gamma \right) + \mathcal{O}\left(\frac{1}{n^2} \right) \right]^2.$$

Proof. The first inequality follows from Theorem 2.2.45 (with an obvious modification). The sum on the right-hand side of the first inequality can be expressed as follows:

$$\sum_{1\le k<m\le n} |l_{k,n}(x)\cdot l_{m,n}(x)| = \left[\left(\sum_{i=1}^{n}|l_{i,n}(x)|\right)^2 - \left(\sum_{i=1}^{n} l_{i,n}^2(x)\right)\right]/2$$
$$= \left[\Lambda_n^2(x) - \left(\sum_{i=1}^{n} l_{i,n}^2(x)\right)\right]/2.$$

In order to estimate the sum $\sum_{i=1}^{n} l_{i,n}^2(x)$, we use the proof of Theorem 2.3 (case $\alpha = 2$) from [64] to get :

$$\sum_{i=1}^{n} l_{i,n}^2(x) \ge c\left(1 + |\cos nt|^2 \sum_{i=1}^{n} i^{-2}\right),$$

where $x = \cos t$ and c is a suitable constant. After some calculation, the sum becomes

$$\sum_{1\le k<m\le n} |l_{k,n}(x)\cdot l_{m,n}(x)| = \frac{\Lambda_n^2(x)}{2} - \frac{c\left(1+(\cos nt)^2\cdot\frac{\pi^2}{6}\right)}{2}.$$

We now use the asymptotic result for the Lebesgue constant $\Lambda_n(X) = \max\limits_{-1\le x\le 1}\Lambda_n(x)$, so we obtain our desired inequality. □

2.2.8 Chebyshev-Grüss inequalities via discrete oscillations for more than two functions

In the last section of the article [2], a Chebyshev-Grüss inequality on a compact metric space for more than two functions was introduced. We obtain a similar result, using the new approach implying discrete oscillations. This result is better than what was obtained in [2] in the sense that the oscillations of the functions are relative only to certain points, while in [2] they are relative to the whole compact metric space X. The results in this section can also be found in [4].

Moreover, in what follows X is an arbitrary set, $B(X)$ the set of all real-valued, bounded functions on X and $f^1,\dots,f^p \in B(X)$. Take $a_n \in \mathbb{R}$, $a_n \ge 0$, $n \ge 0$, such that $\sum_{n=0}^{\infty} a_n = 1$. Furthermore, let $x_n \in X$, $n \ge 0$ be arbitrary mutually distinct points of X. For $f^k \in B(X)$ set $f_n^k := f^k(x_n)$, $k = 1,\dots,p$. Consider a positive linear functional $L : B(X) \to \mathbb{R}$, such that $L(f) := \sum_{n=0}^{\infty} a_n f_n$.

In this section we will use the following notation:

$$osc_L(f^k) := \sup\{|f_n^k - f_m^k| : 0 \le n < m < \infty\}.$$

The following result holds, concerning the oscillations.

Lemma 2.2.68. *Let $B(X)$ be the set of all real-valued and bounded functions on X and let $f^i \in B(X)$, for $i = 1,\dots,p$. Then the following inequality holds*

$$osc_L\Big(\prod_{k=1}^{p} f^k\Big) \le \sum_{i=1}^{p} osc_L(f^i) \prod_{j=1,\, j\ne i}^{p} \sup_{0\le n<\infty}\{|f_n^j|\}.$$

Proof. The above inequality can be proven by induction. If we consider two functions $f^1, f^2 \in B(X)$, we have

$$\begin{aligned}
&\left|f^1(x_n)f^2(x_n) - f^1(x_m)f^2(x_m)\right| \\
&= \left|f^1(x_n)(f^2(x_n) - f^2(x_m)) + (f^1(x_n) - f^1(x_m))f^2(x_m)\right| \\
&\leq \sup_{0\leq k<\infty}\{|f_k^1|\}\left|f^2(x_n) - f^2(x_m)\right| + \sup_{0\leq k<\infty}\{|f_k^2|\}\left|f^1(x_n) - f^1(x_m)\right|.
\end{aligned}$$

We take the supremum on both sides and get

$$osc_L(f^1f^2) \leq osc_L(f^2)\cdot \sup_{0\leq n<\infty}\{|f_n^1|\} + osc_L(f^1)\cdot \sup_{0\leq n<\infty}\{|f_n^2|\}.$$

Now consider the inequality to be true for p and prove it for $p+1$.

$$\begin{aligned}
&osc_L(f^1\cdot f^2 \dots f^p\cdot f^{p+1}) \\
&\leq osc_L(f^1\dots f^p)\cdot \sup_{0\leq n<\infty}\{|f_n^{p+1}|\} + osc_L(f^{p+1})\cdot \sup_{0\leq n<\infty}\{|f_n^1|\}\dots \sup_{0\leq n<\infty}\{|f_n^p|\} \\
&= \sum_{i=1}^{p+1} osc_L(f^i)\prod_{j=1,j\neq i}^{p+1}\sup_{0\leq n<\infty}\{|f_n^j|\}.
\end{aligned}$$

This concludes the proof. $\square$

The next result is a Chebyshev-Grüss inequality via discrete oscillations for more than two functions.

Theorem 2.2.69. *For a positive linear functional $L : B(X) \to \mathbb{R}$, $L(f) := \sum_{n=0}^{\infty} a_n f_n$, $a_n \in \mathbb{R}$, $a_n \geq 0$, $\sum_{n=0}^{\infty} a_n = 1$, the Chebyshev-Grüss-type inequality, involving more than two functions is*

$$\begin{aligned}
&\left|L(f^1\cdot\ldots\cdot f^p) - L(f^1)\cdot\ldots\cdot L(f^p)\right| \\
&\leq \frac{1}{2}\left(1-\sum_{n=0}^{\infty}a_n^2\right)\cdot\sum_{i,j=1,i<j}^{p} osc_L(f^i)\cdot osc_L(f^j)\cdot\prod_{k=1,k\neq i,j}^{p}\sup_{0\leq s<\infty}\{|f_s^k|\}.
\end{aligned}$$

Proof. We prove by induction the following inequality:

$$\begin{aligned}
&\left|L(f^1\cdot\ldots\cdot f^p) - L(f^1)\cdot\ldots\cdot L(f^p)\right| \\
&\leq \sum_{0\leq n<m<\infty} a_n a_m\cdot\sum_{i,j=1,i<j}^{p} osc_L(f^i)\cdot osc_L(f^j)\cdot\prod_{k=1,k\neq i,j}^{p}\sup_{0\leq s<\infty}\{|f_s^k|\}
\end{aligned} \tag{2.2.20}$$

It was proved in [58] that

$$\left|L(f^1\cdot f^2) - L(f^1)\cdot L(f^2)\right| \leq \sum_{0\leq n<m<\infty} a_n a_m\cdot osc_L(f^1)osc_L(f^2),$$

therefore the inequality (2.2.20) is true for $p = 2$. We suppose that the inequality holds for p and we prove it for $p+1$. We have

$$\begin{aligned}
&\left|L(f^1 \cdots f^{p+1}) - L(f^1) \cdots L(f^p)L(f^{p+1})\right| = \\
&\left|L(f^1 \cdots f^{p+1}) - L(f^1 \cdots f^p)L(f^{p+1}) + L(f^1 \dots f^p)L(f^{p+1}) - L(f^1) \cdots L(f^{p+1})\right| \\
&\leq \sum_{0 \leq n < m < \infty} a_n a_m \cdot osc_L(f^1 \dots f^p) \cdot osc_L(f^{p+1}) \\
&+ \left|L(f^1 f^2 \dots f^p) - L(f^1) \dots L(f^p)\right| \cdot \sup_{0 \leq s < \infty} \{|f_s^{p+1}|\} \\
&\leq \sum_{0 \leq n < m < \infty} a_n a_m \cdot \left(\sum_{i=1}^{p} osc_L(f^i) \prod_{j=1, j \neq i}^{p} \sup_{0 \leq s < \infty} \{|f_s^j|\} \right) \cdot osc_L(f^{p+1}) \\
&+ \sum_{0 \leq n < m < \infty} a_n a_m \cdot \left(\sum_{i,j=1; i<j}^{p} osc_L(f^i) \cdot osc_L(f^j) \cdot \prod_{k=1; k \neq i,j}^{p} \sup_{0 \leq n < \infty} \{|f_n^k|\} \right) \cdot \sup_{0 \leq s < \infty} \{|f_s^{p+1}|\} \\
&= \sum_{0 \leq n < m < \infty} a_n a_m \sum_{i,j=1; i<j}^{p+1} osc_L(f^i) \cdot osc_L(f^j) \cdot \prod_{k=1; k \neq i,j}^{p+1} \sup_{0 \leq s < \infty} \{|f_s^k|\}.
\end{aligned}$$

Using in (2.2.20) the following identity

$$\sum_{0 \leq n < m < \infty} a_n a_m = \frac{1}{2} \left(1 - \sum_{n=0}^{\infty} a_n^2 \right),$$

the theorem is proved. $\square$

3 Bivariate Chebyshev-Grüss Inequalities on compact metric spaces

3.1 Auxiliary and historical results

In this section we recall a method for approximating functions defined on the product of two compact metric spaces (X, d_X) and (Y, d_Y). We assume that $X \times Y$ is endowed with the product topology generated by the two compact metric spaces. For references about parametric extensions of univariate operators, see the papers [86], [108], [63], [78] and [45].

Consider the compact metric spaces X and Y. We define $\hat{x} : C(X) \to \mathbb{R}$ by $\hat{x}(f) = f(x)$, and $\hat{x} : C(X \times Y) \to C(Y)$ by $(\hat{x}f)(y) = f(x,y)$. In a similar way, we have $\hat{y} : C(Y) \to \mathbb{R}$ defined by $\hat{y}(f) = f(y)$ and $\hat{y} : C(X \times Y) \to C(X)$ by $(\hat{y}f)(x) = f(x,y)$. Now let $L_1 : C(X) \to C(X)$ be a linear operator and define the parametric extension of L_1 by $\overline{L}_1 : C(X \times Y) \to C(X \times Y)$, such that

$$(\overline{L}_1 f)(x,y) := \hat{x} L_1 \hat{y} f.$$

Similarly, we get the parametric extension of a linear operator $L_2 : C(Y) \to C(Y)$, $\overline{L}_2 : C(X \times Y) \to C(X \times Y)$ defined by

$$(\overline{L}_2 f)(x,y) := \hat{y} L_2 \hat{x} f.$$

The above approach can be found in [26], [29] and [45].

Some estimates in terms of $\widetilde{\omega}_{d_{X\times Y}}$ are given now. We consider, just like before, that $X \times Y$ carries the product topology of the spaces (X, d_X) and (Y, d_Y). This topology can be generated by a multitude of metrics, for example by

$$d_p((x,y),(\hat{x},\hat{y})) := (d_X(x,\hat{x})^p + d_Y(y,\hat{y})^p)^{\frac{1}{p}}, 1 \le p < \infty,$$

and by

$$d_\infty((x,y),(\hat{x},\hat{y})) := \max\{d_X(x,\hat{x}), d_Y(y,\hat{y})\}.$$

The Euclidean metric will mostly be used in the sequel. For $p = 2$ in the first example of metrics from above, we have

$$d_2((x,y),(\hat{x},\hat{y})) := \left(d_X(x,\hat{x})^2 + d_Y(y,\hat{y})^2\right)^{\frac{1}{2}}.$$

Whenever we have estimates for continuous functions $f \in C(X \times Y)$, in terms of $\omega_{d_{X\times Y}}$, where $d_{X\times Y}$ is a metric on the compact space $X \times Y$, these estimates will depend upon the concrete metric $d_{X\times Y}$. We suppose also in the bivariate case that our operators reproduce the constant functions 1_X and 1_Y.

In [45] an inequality in terms of $\widetilde{\omega}_{d_{X\times Y}}$ was given.

Theorem 3.1.1. *(see Theorem 6.2 in [45]) Let X and Y given as before, and let $L_1 : C(X) \to C(X)$ and $L_2 : C(Y) \to C(Y)$ be positive linear operators reproducing constant functions. Let $d_{X\times Y}$ be a metric on $X \times Y$ such that*

$$d_{X\times Y}((x,r),(\hat{x},r)) = d_X(x,\hat{x}) \text{ for all } r \in Y \text{ and for all } (x,\hat{x}) \in X^2,$$

and

$$d_{X\times Y}((s,y),(s,\hat{y})) = d_Y(y,\hat{y}) \text{ for all } s \in X \text{ and for all } (y,\hat{y}) \in Y^2.$$

Then for $f \in C(X \times Y)$, $(x,y) \in X \times Y$, and the product $\overline{L}_1 \circ \overline{L}_2$ of parametric extensions, we have for any $\epsilon > 0$

$$\begin{aligned}&\left|(I - (\overline{L}_1 \circ \overline{L}_2))(f;x,y)\right| \\ &\leq \max\{1, \epsilon^{-1} \cdot (L_1(d_X(\cdot,x);x) + L_2(d_Y(\cdot,y);y))\} \cdot \widetilde{\omega}_{d_{X\times Y}}(f;\epsilon).\end{aligned}$$

For the case $X \times Y = [a,b] \times [c,d]$ we have

Corollary 3.1.2. *Let $X \times Y = [a,b] \times [c,d]$, and let the metrics on $[a,b]$ and $[c,d]$ be given by $d(x,y) = |x-y|$. Then under the assumptions of the above theorem, we have the following. For $1 \leq q \leq \infty$ and for any $\epsilon > 0$ it holds:*

$$\begin{aligned}&\left|(I - (\overline{L}_1 \circ \overline{L}_2))(f;x,y)\right| \\ &\leq \max\{1, \epsilon^{-1} \cdot (L_1(d_X(\cdot,x);x) + L_2(d_Y(\cdot,y);y))\} \cdot \widetilde{\omega}_{d_q}(f;\epsilon).\end{aligned}$$

Here $\widetilde{\omega}_{d_q}(f;\epsilon)$ is the least concave majorant of the modulus of continuity.

Remark 3.1.3. i) One can see that in the above corollary, the quantity in front of $\widetilde{\omega}_{d_q}(f;\epsilon)$ does not depend on q.

ii) If in the above theorem we choose $\epsilon = L_1(d_X(\cdot,x);x) + L_2(d_Y(\cdot,y);y)$ (if this quantity is > 0), then we obtain

$$\begin{aligned}&\left|(I - (\overline{L}_1 \circ \overline{L}_2))(f;x,y)\right| \\ &\leq \widetilde{\omega}_{d_{X\times Y}}(f; L_1(d_X(\cdot,x);x) + L_2(d_Y(\cdot,y);y)) \\ &\leq \widetilde{\omega}_{d_{X\times Y}}(f; L_1(d_X(\cdot,x);x)) + \widetilde{\omega}_{d_{X\times Y}}(f; L_2(d_Y(\cdot,y);y)).\end{aligned}$$

3.2 Bivariate (positive) linear operators

3.2.1 Bivariate Bernstein operators

Let $X = [0,1]$ and $I = [0,1] \times [0,1]$ a compact metric space endowed with the Euclidean metric

$$d_2((s,t),(x,y)) := \sqrt{(s-x)^2 + (t-y)^2},$$

for $(s,t),(x,y) \in I$. The bivariate Bernstein operators, introduced by P. L. Butzer in [18], are given by

$$B_{n_1,n_2}(f;(x,y)) := \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}\right) b_{n_1,k_1}(x) b_{n_2,k_2}(y),\ f \in \mathbb{R}^I,\ x,y \in X,$$

where $b_{n_1,k_1}(x) := \binom{n_1}{k_1} x^{k_1}(1-x)^{n_1-k_1}$ and $b_{n_2,k_2}(y) := \binom{n_2}{k_2} y^{k_2}(1-y)^{n_2-k_2}$.

The second moment of the bivariate Bernstein polynomial in this case is given by

$$\begin{aligned} B_{n_1,n_2}(d_2^2(\cdot,(x,y));(x,y)) &= \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} d_2^2\left(\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right),(x,y)\right) b_{n_1,k_1}(x) b_{n_2,k_2}(y) \\ &= \frac{x(1-x)}{n_1} + \frac{y(1-y)}{n_2} \le \frac{1}{4}\left(\frac{1}{n_1}+\frac{1}{n_2}\right), \end{aligned}$$

for

$$d_2^2\left(\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right),(x,y)\right) = \left(\frac{k_1}{n_1}-x\right)^2 + \left(\frac{k_2}{n_2}-y\right)^2.$$

3.2.2 Products of King operators

Let $r_{n_1}(x)$, $r_{n_2}(y)$ be sequences of continuous functions with $0 \le r_{n_1}(x) \le 1$ and $0 \le r_{n_2}(y) \le 1$. The bivariate King operator $V_{n_1,n_2} : C([0,1]\times[0,1]) \to C([0,1]\times[0,1])$, $n_1 \ne n_2$ can be defined by

$$V_{n_1,n_2}(f;(x,y)) := \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} v_{n_1,k_1}(x)\cdot v_{n_2,k_2}(y)\cdot f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right),$$

with

$$\begin{aligned} v_{n_1,k_1}(x) &= \binom{n_1}{k_1}(r_{n_1}(x))^{k_1}(1-r_{n_1}(x))^{n_1-k_1}, \\ v_{n_2,k_2}(y) &= \binom{n_2}{k_2}(r_{n_2}(y))^{k_2}(1-r_{n_2}(y))^{n_2-k_2}, \end{aligned}$$

for $f \in C([0,1]\times[0,1])$, $0 \le x,y \le 1$. v_{n_1,k_1} and v_{n_2,k_2} are fundamental functions of the V_{n_1,n_2} operator.

Remark 3.2.1. For $r_{n_1}(x) = x$, $r_{n_2}(y) = y$, $n_1, n_2 = 1,2,\ldots$, the positive linear operators V_{n_1,n_2} reduce to the bivariate Bernstein operators.

We now give some properties of the above defined operators, that are generalizations of those from the univariate case.

Proposition 3.2.2.

i) The operators V_{n_1} and V_{n_2} reproduce constants, so V_{n_1,n_2} also reproduces them.

ii) If we consider the projection maps, i.e., $pr_1(x,y) = x$ and $pr_2(x,y) = y$, then we have

$$\begin{aligned} V_{n_1,n_2}(pr_1;(x,y)) &= r_{n_1}(x) \\ V_{n_1,n_2}(pr_2;(x,y)) &= r_{n_2}(y). \end{aligned}$$

iii) For the function $e(x,y) = x^2 + y^2$, we get

$$\begin{aligned} &V_{n_1,n_2}(e;(x,y)) = V_{n_1}(e_2;x) + V_{n_2}(e_2;y) \\ &= \left(\frac{r_{n_1}(x)}{n_1} + \frac{n_1-1}{n_1}(r_{n_1}(x))^2\right) + \left(\frac{r_{n_2}(y)}{n_2} + \frac{n_2-1}{n_2}(r_{n_2}(y))^2\right). \end{aligned}$$

iv) $\lim\limits_{n_1,n_2\to\infty} V_{n_1,n_2}(f;(x,y)) = f(x,y)$, *for each* $f \in C([0,1]\times[0,1])$, $x,y\in[0,1]$, *if and only if*

$$\lim_{n_1\to\infty} r_{n_1}(x) = x$$

and

$$\lim_{n_2\to\infty} r_{n_2}(y) = y.$$

v) The second moment in the general bivariate case is given by

$$V_{n_1,n_2}(d_2^2(\cdot,(x,y));(x,y)) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} d_2^2\left(\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right),(x,y)\right) v_{n_1,k_1}(x)v_{n_2,k_2}(y)$$
$$= \frac{r_{n_1}(x)}{n_1}[1-r_{n_1}(x)] + [r_{n_1}(x)-x]^2 + \frac{r_{n_2}(y)}{n_2}[1-r_{n_2}(y)] + [r_{n_2}(y)-y]^2,$$

where $0 \le r_{n_1}(x) \le 1$ *and* $0 \le r_{n_2}(y) \le 1$ *are continuous functions and for*

$$d_2^2\left(\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right),(x,y)\right) = \left(\frac{k_1}{n_1}-x\right)^2 + \left(\frac{k_2}{n_2}-y\right)^2.$$

Again the interest is to find a sequence of positive linear operators defined on $C([0,1]\times[0,1])$ that reproduce both constant and e functions, defined as above. For this we need special choices of $r_{n_1}(x) = r^*_{n_1}(x)$ and $r_{n_2}(y) = r^*_{n_2}(y)$. We have the following result.

Theorem 3.2.3. *Let* $\{V_{n_1,n_2}\}_{n_1,n_2\in\mathbb{N}}$ *be the sequence of operators defined above, with*

$$r^*_{n_1}(x) := \begin{cases} r^*_1(x) = x^2 & ,\textit{for } n_1 = 1 \\ r^*_{n_1}(x) = -\frac{1}{2(n_1-1)} + \sqrt{\frac{n_1}{n_1-1}x^2 + \frac{1}{4(n_1-1)^2}} & ,\textit{for } n_1 = 2,3,\ldots, \end{cases}$$

and the same holding for $r^*_{n_2}(y)$. *Then we get*

$$V^*_{n_1,n_2}(e;(x,y)) = V^*_{n_1}(e_2;x) + V^*_{n_2}(e_2;y) = x^2+y^2,$$

for $n_1,n_2\in\mathbb{N}$, $n_1 \ne n_2$ *and* $x,y\in[0,1]$. *Also,* $V^*_{n_1,n_2}(pr_1;(x,y)) \ne pr_1(x,y) = x$ *and* $V^*_{n_1,n_2}(pr_2;(x,y)) \ne pr_2(x,y) = y$. $V^*_{n_1,n_2}$ *is also not a polynomial operator.*

If we talk about the fundamental functions of these operators, given by

$$v^*_{n_1,k_1}(x) = \binom{n_1}{k_1}(r^*_{n_1}(x))^{k_1}(1-r^*_{n_1}(x))^{n_1-k_1}$$

and the same for $v^*_{n_2,k_2}(x)$, then they satisfy $\sum_{k_1=0}^{n_1} v^*_{n_1,k_1}(x) = 1$ and $\sum_{k_2=0}^{n_2} v^*_{n_2,k_2}(y) = 1$ for $n_1,n_2 = 1,2,\ldots$.

Proposition 3.2.4 (Properties of $r^*_{n_1}, r^*_{n_2}$).

i) $0 \le r^*_{n_1}(x) \le 1$ *and* $0 \le r^*_{n_2}(y) \le 1$, *for* $n_1,n_2 = 1,2,\ldots$, *and* $0 \le x,y \le 1$.

ii) $\lim_{n_1\to\infty} r^*_{n_1}(x) = x$ *and* $\lim_{n_2\to\infty} r^*_{n_2}(y) = y$, *for* $0 \le x,y \le 1$.

The second moment of the bivariate special King operators $V^*_{n_1,n_2}$ is:

$$V^*_{n_1,n_2}(d_2^2(\cdot,(x,y));(x,y)) = 2x(x - r^*_{n_1}(x)) + 2y(y - r^*_{n_2}(y)). \tag{3.2.1}$$

We can discriminate between more cases, but the interesting one is for $n_1, n_2 = 2, 3, \ldots, n_1 \neq n_2$. Then we have

$$r^*_{n_1}(x) = -\frac{1}{2(n_1-1)} + \sqrt{\left(\frac{n_1}{n_1-1}\right)x^2 + \frac{1}{4(n_1-1)^2}}, \tag{3.2.2}$$

and for $r^*_{n_2}(y)$ we have an analogous relation.

If we replace (3.2.2) in (3.2.1), we get the second moment for this special case.

We now want to find r_{n_1} and r_{n_2}, such that the second moment in the general case $V_{n_1,n_2}(d_2^2(\cdot,(x,y));(x,y))$ is minimal. We have seen in the univariate case that

$$r^{min}_{n_1}(x) := \begin{cases} 0 & , x \in [0, \frac{1}{n_1}), \\ \frac{2n_1x-1}{2n_1-2} & , x \in [\frac{1}{2n_1}, 1 - \frac{1}{2n_1}], \\ 1 & , x \in (1 - \frac{1}{2n_1}, 1], \end{cases}$$

and something similar holds for $r^{min}_{n_2}(y)$. Using these functions, we get the minimum value of the second moment. For this, we have the following representation

$$V^{min}_{n_1,n_2}((\cdot - (x,y))^2;(x,y)) := V^{min}_{n_1}((e_1 - x)^2;x) + V^{min}_{n_2}((e_1 - y)^2;y),$$

where $V^{min}_{n_1}((e_1 - x)^2;x)$ and $V^{min}_{n_2}((e_1 - y)^2;y)$ are given as in the univariate case.

3.2.3 Products of Hermite-Fejér interpolation operators

We consider tensor products of two parametric extensions of classical univariate Hermite-Fejér interpolation operators defined with respect to Chebyshev roots of the first kind $x_k = \cos\left(\frac{2k-1}{2n}\pi\right)$, $1 \leq k \leq n$, and given by

$$H_{2n-1}(f;x) := \sum_{k=1}^{n} f(x_k) \cdot (1 - x \cdot x_k) \cdot \left(\frac{T_n(x)}{n(x - x_k)}\right)^2,$$

for $f \in \mathbb{R}^{[-1,1]}$ and with $T_n(x) = \cos(n \cdot \arccos(x))$ the n-th Chebyshev polynomial of the first kind.

The tensor product of two parametric extensions of univariate Hermite-Fejér operators is given by

$$\begin{aligned} &H_{2n_1-1,2n_2-1}(f;x,y) \\ &:= \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} f(x_{k_1}, y_{k_2})(1 - x \cdot x_{k_1}) \cdot (1 - y \cdot y_{k_2}) \cdot \left(\frac{T_{n_1}(x)}{n_1(x - x_{k_1})}\right)^2 \cdot \left(\frac{T_{n_2}(y)}{n_2(y - y_{k_2})}\right)^2, \end{aligned}$$

for $n_i \geq 1$, $i = 1, 2$, $f \in C([-1,1]^2)$ and all $(x,y) \in I = [-1,1]^2$.

In [45], H. Gonska showed that for such Hermite-Fejér operators, the following inequality involving the first absolute moment holds:

$$H_{2n-1}(|e_1 - x|;x) \leq \frac{4}{n} \cdot |T_n(x)| \cdot \{\sqrt{1-x^2} \cdot \ln n + 1\} \leq 10\,|T_n(x)| \cdot \frac{\ln n}{n}.$$

As one can see from the proof of the above result (see Lemma 6.9 in [45]), we can say that the first absolute moment of the bivariate operators is given by

$$
\begin{aligned}
&H_{2n_1-1,2n_2-1}\left(d_1(\cdot,(x,y));(x,y)\right)\\
&= H_{2n_1-1}(|\cdot_1 - x|\,;x) + H_{2n_2-1}(|\cdot_2 - y|\,;y)\\
&\leq \frac{4}{n_1}\cdot|T_{n_1}(x)|\cdot\{\sqrt{1-x^2}\cdot\ln n_1+1\} + \frac{4}{n_2}\cdot|T_{n_2}(y)|\cdot\{\sqrt{1-y^2}\cdot\ln n_2+1\}\\
&= 4\cdot\left\{\left(\frac{1}{n_1}\cdot|T_{n_1}(x)|\cdot\{\sqrt{1-x^2}\cdot\ln n_1+1\}\right) + \left(\frac{1}{n_2}\cdot|T_{n_2}(y)|\cdot\{\sqrt{1-y^2}\cdot\ln n_2+1\}\right)\right\},
\end{aligned}
$$

if considering the metric

$$d_1((s,t),(x,y)) = |s-x| + |t-y|\,.$$

Then we can now recall another result from [45] (see Proposition 6.19 there).

Proposition 3.2.5. *If $H_{2n_1-1,2n_2-1}$, $n_i \geq 1$, $i=1,2$, denotes the product of two parametric extensions of univariate Hermite-Fejér operators H_{2n_1-1} and H_{2n_2-1}, both based on the roots of Chebyshev polynomials T_{n_1} and T_{n_2}, respectively, then for all $f \in C([-1,1]^2)$ and all $(x,y)\in[-1,1]^2$ there holds*

$$
\begin{aligned}
&|(I - H_{2n_1-1,2n_2-1})(f;x,y)|\\
&\leq 4\cdot\tilde{\omega}_{d_1}\left(f;n_1^{-1}\cdot|T_{n_1}(x)|\cdot\{\sqrt{1-x^2}\cdot\ln n_1+1\} + n_2^{-1}\cdot|T_{n_1}(y)|\cdot\{\sqrt{1-y^2}\cdot\ln n_2+1\}\right),
\end{aligned}
$$

where I denotes the identity mapping on $C([-1,1]^2)$.

The second moment of the bivariate Hermite-Fejér operator can also be evaluated. If we now consider

$$d_2((s,t),(x,y)) := \sqrt{(s-x)^2+(t-y)^2},$$

we have

$$H_{2n_1-1,2n_2-1}(d_2^2(\cdot,(x,y));(x,y)) = \frac{1}{n_1}\cdot T_{n_1}^2(x) + \frac{1}{n_2}\cdot T_{n_2}^2(y).$$

3.2.4 Products of quasi-Hermite-Fejér interpolation operators

We consider tensor products of two parametric extensions of univariate quasi-Hermite-Fejér interpolation operators defined with respect to Chebyshev roots of the second kind $x_v = \cos\left(\frac{v}{n+1}\pi\right)$, $1 \leq v \leq n$, and given by

$$Q_n(f;x) := \sum_{v=0}^{n+1} f(x_v)\cdot F_{n,v}(x)\cdot U_n^2(x),$$

for $f \in \mathbb{R}^{[-1,1]}$ and $U_n(x)$ the n-th Chebyshev polynomial of the second kind, where

$$F_{n,v}(x) := \begin{cases} \frac{1-x}{2(n+1)^2} & \text{, for } v=0,\\ \frac{(1-x^2)(1-x_v\cdot x)}{(n+1)^2(x-x_v)^2} & \text{, for } 1\leq v\leq n,\\ \frac{1+x}{2(n+1)^2} & \text{, for } v=n+1. \end{cases}$$

The tensor product of two parametric extensions of univariate quasi-Hermite-Fejér operators is given by

$$Q_{n_1,n_2}(f;x,y) := \sum_{v_1=0}^{n_1+1}\sum_{v_2=0}^{n_2+1} f(x_{v_1},y_{v_2}) \cdot F_{n_1,v_1}(x) \cdot F_{n_2,v_2}(y) \cdot U_{n_1}^2(x) \cdot U_{n_2}^2(y)$$

for $n_i \geq 1$, $i = 1,2$, $f \in C([-1,1]^2)$ and all $(x,y) \in I = [-1,1]^2$. We have

$$F_{n_1,v_1}(x) := \begin{cases} \frac{1-x}{2(n_1+1)^2} & \text{, for } v_1 = 0 \\ \frac{(1-x^2)(1-x_{v_1}\cdot x)}{(n_1+1)^2\cdot(x-x_{v_1})^2} & \text{, for } 1 \leq v_1 \leq n_1 \\ \frac{1+x}{2(n_1+1)^2} & \text{, for } v_1 = n_1+1 \end{cases}$$

and a similar definition for $F_{n_2,v_2}(y)$.

The first absolute moment of the bivariate operators is given by

$$\begin{aligned} & Q_{n_1,n_2}\left(d_1(\cdot,(x,y));(x,y)\right) \\ & = Q_{n_1}(|\cdot_1 - x|\,;x) + Q_{n_2}(|\cdot_2 - y|\,;y) \\ & \leq (1-x^2)\,|U_{n_1}(x)| \cdot \frac{10\ln(n_1+1)}{n_1+1} + (1-y^2)\,|U_{n_2}(y)| \cdot \frac{10\ln(n_2+1)}{n_2+1}. \end{aligned}$$

The second moment of the bivariate quasi-Hermite-Fejér operator can also be evaluated. We have

$$\begin{aligned} & Q_{n_1,n_2}(d_2^2(\cdot,(x,y));(x,y)) \\ & = Q_{n_1}((\cdot_1 - x)^2;x) + Q_{n_2}((\cdot_2 - y)^2;y) \\ & = \frac{1}{n_1+1} \cdot (1-x^2) \cdot U_{n_1}^2(x) + \frac{1}{n_2+1} \cdot (1-y^2) \cdot U_{n_2}^2(y). \end{aligned}$$

3.2.5 Products of almost-Hermite-Fejér interpolation operators

We consider tensor products of two parametric extensions of univariate almost-Hermite-Fejér interpolation operators given by

$$F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}(f;x) := \sum_{v=0}^{n} f(x_v) \cdot E_{v,n}(x),$$

where l_v is the vth Lagrange fundamental polynomial, $x_v = \cos\frac{2v}{2n+1}\pi$, $1 \leq v \leq n$ and

$$w(x) = \frac{\sin\frac{2n+1}{2}\arccos x}{\sin\frac{1}{2}\arccos x}.$$

In the above definition, we have

$$E_{v,n}(x) := \begin{cases} \frac{w^2(x)}{w^2(1)} & \text{, for } v = 0, \\ \frac{1-x}{1-x_v} \cdot \frac{1-xx_v}{1-x_v^2} \cdot l_v^2(x) & \text{, for } 1 \leq v \leq n. \end{cases}$$

The tensor product of two parametric extensions of univariate almost-Hermite-Fejér operators is given by

$$F_{1,0;n_1,n_2}^{(\frac{1}{2},-\frac{1}{2})}(f;x,y) := \sum_{v_1=1}^{n_1}\sum_{v_2=1}^{n_2} f(x_{v_1},y_{v_2}) \cdot E_{v_1,n_1}(x) \cdot E_{v_2,n_2}(y),$$

where

$$E_{v_1,n_1}(x) := \begin{cases} \frac{w(x)^2}{w(1)^2} & \text{, for } v_1 = 0, \\ \frac{1-x}{1-x_{v_1}} \cdot \frac{1-x\cdot x_{v_1}}{1-x_{v_1}^2} \cdot l_{v_1}^2(x) & \text{, for } 1 \le v_1 \le n_1 \end{cases}$$

and

$$E_{v_2,n_2}(y) := \begin{cases} \frac{w(y)^2}{w(1)^2} & \text{, for } v_2 = 0, \\ \frac{1-y}{1-y_{v_2}} \cdot \frac{1-y\cdot y_{v_2}}{1-y_{v_2}^2} \cdot l_{v_2}^2(y) & \text{, for } 1 \le v_2 \le n_2. \end{cases}$$

The first absolute moment of the bivariate operators is given by

$$\begin{aligned}
&F_{1,0;n_1,n_2}^{(\frac{1}{2},-\frac{1}{2})}(d_1(\cdot,(x,y));(x,y)) \\
&= F_{1,0;n_1}^{(\frac{1}{2},-\frac{1}{2})}(|\cdot_1 - x|\,;x) + F_{1,0;n_2}^{(\frac{1}{2},-\frac{1}{2})}(|\cdot_2 - y|\,;y) \\
&\le c_1 \cdot \frac{1+\sqrt{1-x^2}\ln(n_1)}{2n_1+1} + c_2 \cdot \frac{1+\sqrt{1-y^2}\ln(n_2)}{2n_2+1},
\end{aligned}$$

for suitable constants c_1, c_2.

The second moment of the bivariate almost-Hermite-Fejér operator can also be expressed. We have

$$\begin{aligned}
&F_{1,0;n_1,n_2}^{(\frac{1}{2},-\frac{1}{2})}(d_2^2(\cdot,(x,y));(x,y)) \\
&= F_{1,0;n_1}^{(\frac{1}{2},-\frac{1}{2})}((\cdot_1 - x)^2;x) + F_{1,0;n_2}^{(\frac{1}{2},-\frac{1}{2})}((\cdot_2 - y)^2;y) \\
&= \frac{2(1-x)\cdot w(x)^2}{3n_1} + \frac{2(1-y)\cdot w(y)^2}{3n_2} \\
&= \frac{2}{3}\cdot\left(\frac{(1-x)\cdot w(x)^2}{n_1} + \frac{(1-y)\cdot w(y)^2}{n_2}\right).
\end{aligned}$$

3.2.6 Products of convolution operators

Take $X = [-1,1]$ and $C(X)$ the space of real-valued continuous functions defined on X. The convolution operator $G_{m(n)}$ is defined as before. If we consider a bivariate function defined on $[-1,1] \times [-1,1] = I$, then the parametric extensions of the operator $G_{m(n)}f$ are given by

$$\begin{aligned}
({}_xG_{m(n_1)}f)(x,y) &:= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\cos(arccos(x)+\nu_1),y)\cdot K_{m(n_1)}(\nu_1)\mathrm{d}\nu_1, \\
({}_yG_{m(n_2)}f)(x,y) &:= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,\cos(arccos(y)+\nu_2))\cdot K_{m(n_2)}(\nu_2)\mathrm{d}\nu_2,
\end{aligned}$$

where the kernels $K_{m(n_1)}, K_{m(n_2)}$ are positive and even trigonometric polynomials of degrees $m(n_1)$ and $m(n_2)$, satisfying

$$\int_{-\pi}^{\pi} K_{m(n_i)}(\nu_i)\mathrm{d}\nu_i = \pi,\ i = 1,2,$$

meaning that $G_{m(n_1)}(1,x) = 1$ and $G_{m(n_2)}(1,y) = 1$, for $x, y \in [-1,1]$. Both $G_{m(n_i)}(f,\cdot)$, $i = 1,2$ are algebraic polynomials of degree $m(n_i)$, $i = 1,2$ and the kernel $K_{m(n_i)}$ has the form:

$$K_{m(n_i)}(\nu_i) = \frac{1}{2} + \sum_{k_i=1}^{m(n_i)} \rho_{k_i,m(n_i)} \cdot \cos(k_i \nu_i),\ i = 1,2,$$

for $\nu_i \in [-\pi, \pi]$.

Since we know that $G_{m(n)}$ is a positive linear operator, we have that its parametric extensions ${}_xG_{m(n_1)}$ and ${}_yG_{m(n_2)}$ are also positive linear operators.

Proposition 3.2.6. *The parametric extensions ${}_xG_{m(n_1)}$, ${}_yG_{m(n_2)}$ satisfy the relation*

$${}_xG_{m(n_1)} \cdot_y G_{m(n_2)} =_y G_{m(n_2)} \cdot_x G_{m(n_1)}.$$

Their product is the bidimensional operator $G_{m(n_1),m(n_2)}$, which, for every function $f \in C(I)$ looks as:

$$\begin{aligned}&G_{m(n_1),m(n_2)}(f;(x,y))\\&= \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\cos(arccos(x)+\nu_1), \cos(arccos(y)+\nu_2)) \cdot K_{m(n_1)}(\nu_1) \cdot K_{m(n_2)}(\nu_2) \mathrm{d}\nu_1 \mathrm{d}\nu_2.\end{aligned}$$

For different degrees $m(n_i),\ i = 1,2$, we get different convolution operators and different second moments, respectively. This is why we talk about second moments in a subsequent section.

3.2.7 Bivariate Shepard operators

Let us take the domain $I = [0,1] \times [0,1]$ and consider the metric

$$d_1((s,t),(x,y)) = |s-x| + |t-y|,$$

for $(s,t),(x,y) \in I$. We then obtain a first special case of the CBS operator. If we have a function $f : [0,1] \times [0,1] \to \mathbb{R}$ and consider the bivariate CBS operator based on equidistant pairs of points as a tensor product of two univariate operators (see [39]), we get

$$(S^{\mu_1,\mu_2}_{n_1+1,n_2+1})(f;x,y) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} s_{k_1,\mu_1}(x) s_{k_2,\mu_2}(y) f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}\right),$$

where $\mu_1, \mu_2 > 1$ and with

$$s_{k_1,\mu_1}(x) = \frac{\left|x - \frac{k_1}{n_1}\right|^{-\mu_1}}{\sum\limits_{k_1=0}^{n_1}\left|x - \frac{k_1}{n_1}\right|^{-\mu_1}},$$

$$s_{k_2,\mu_2}(y) = \frac{\left|y - \frac{k_2}{n_2}\right|^{-\mu_2}}{\sum\limits_{k_2=0}^{n_2}\left|y - \frac{k_2}{n_2}\right|^{-\mu_2}}.$$

For the univariate CBS operators $S_{n_1+1}^{\mu_1}(f;x)$, based on n_1+1 equidistant points and $S_{n_2+1}^{\mu_2}(f;y)$, based on n_2+1 equidistant points, with respect to a univariate function f, we have

$$S_{n_1+1,n_2+1}^{\mu_1,\mu_2}(f;x,y) = S_{n_1+1}^{\mu_1}(f;x) \cdot S_{n_2+2}^{\mu_2}(f;y).$$

The first absolute moment of the tensor product CBS operator given above is

$$S_{n_1+1,n_2+1}^{\mu_1,\mu_2}(d_1(\cdot,(x,y));(x,y)) = S_{n_1+1}^{\mu_1}(|\cdot - x|\,;x) + S_{n_2+1}^{\mu_2}(|\cdot - y|\,;y).$$

A proof for this will be given later on (see the proof of Theorem 3.3.11).

For this special case, we will give a pre-Chebyshev-Grüss inequality in one of the sections to follow (see Theorem 3.3.11).

If we now consider the compact metric space $I = [0,1] \times [0,1]$ endowed with the Euclidean metric, we give another bivariate CBS operator as a tensor product of two CBS operators defined as in (1.2.3):

$$S_{n_1,n_2}^{\mu_1,\mu_2}(f;(x,y)) := \begin{cases} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} f(x_{k_1},y_{k_2}) \cdot s_{k_1}^{\mu_1}(x) \cdot s_{k_2}^{\mu_2}(y), & x \notin \{x_1,\dots,x_{n_1}\}, \\ & y \notin \{y_1,\dots,y_{n_2}\}, \\ f(x_{k_1},y_{k_2}), & \text{otherwise.} \end{cases}$$

We can give a relation for the second moment of this bivariate CBS operator $S_{n_1,n_2}^{\mu_1,\mu_2}$. It holds

$$S_{n_1,n_2}^{\mu_1,\mu_2}(d_2^2(\cdot,(x,y));(x,y)) = \begin{cases} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} d_2^2(\cdot;(x,y)) \cdot s_{k_1}^{\mu_1}(x) \cdot s_{k_2}^{\mu_2}(y), & x \notin \{x_1,\dots,x_{n_1}\}, \\ & y \notin \{y_1,\dots,y_{n_2}\}, \\ 0, & \text{otherwise.} \end{cases}$$

For the second moment of this CBS operator $S_{n_1,n_2}^{\mu_1,\mu_2}$ we also have an equality similar to the one for the first absolute moment:

$$S_{n_1,n_2}^{\mu_1,\mu_2}(d_2^2(\cdot,(x,y));(x,y)) = S_{n_1}^{\mu_1}((e_1-x)^2;x) + S_{n_2}^{\mu_2}((e_1-y)^2;y).$$

We will consider different cases, meaning for $\mu_1 = \mu_2 = 1$, $1 < \mu_1 = \mu_2 < 2$, $\mu_1 = \mu_2 = 2$ and $\mu_1 = \mu_2 > 2$. For these cases we will have different second and first absolute moments and different (pre-) Chebyshev-Grüss inequalities, respectively.

Remark 3.2.7. The original bivariate operators that were considered by Shepard in [104] were not constructed as tensor products of unvariate operators. For more details about other ways of describing such Shepard operators, see [39]. They look as follows

$$S_n^\mu f(x,y) := \sum_{i=0}^{n} s_{n,i}^{(\mu)}(x,y) f(x_i,y_i), \text{ if } (x,y) \neq (x_i,y_i),$$

$S_n^\mu f(x_i,y_i) = f(x_i,y_i)$, where $\mu > 0$ fixed, $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^2$, $(x_i,y_i) \in D$, $i = 0,\dots,n$, $x_0 < x_1 < \dots < x_n$, $y_0 < y_1 < \dots < y_n$,

$$s_{n,i}^{(\mu)}(x,y) = \left[(x-x_i)^2 + (y-y_i)^2\right]^{-\frac{\mu}{2}} / l_n^{(\mu)}(x,y),$$

$$l_n^{(\mu)}(x,y) = \sum_{i=0}^{n} \left[(x-x_i)^2 + (y-y_i)^2\right]^{-\frac{\mu}{2}}.$$

Such Shepard operators are used in Computer Aided Geometric Design. Nevertheless, because of the scarce distribution of the points (x_i,y_i), for $i = 0,\dots,n$ in D, the convergence properties we are interested in are relatively poor.

3.2.8 Products of piecewise linear interpolation operators at equidistant knots

Let $X = [0,1]$ and $I = [0,1] \times [0,1]$ be a compact metric space, together with the Euclidean metric.

The bivariate piecewise linear interpolation operator at equidistant knots $S_{\Delta_{n_1},\Delta_{n_2}}$: $C(I) \to C(I)$ at the points $0, \frac{1}{n_1} \dots, \frac{k_1}{n_1}, \dots, \frac{n_1-1}{n_1}, 1$ and $0, \frac{1}{n_2} \dots, \frac{k_2}{n_2}, \dots, \frac{n_2-1}{n_2}, 1$, respectively, can be explicitely described as

$$S_{\Delta_{n_1},\Delta_{n_2}}(f;(x,y)) =$$
$$\frac{1}{n_1 n_2}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\left[\frac{k_1-1}{n_1},\frac{k_1}{n_1},\frac{k_1+1}{n_1};|\alpha-x|\right]_\alpha\left[\frac{k_2-1}{n_2},\frac{k_2}{n_2},\frac{k_2+1}{n_2};|\alpha-y|\right]_\alpha f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right).$$

Denote by

$$u_{n_1,k_1}(x) = \frac{1}{n_1}\left[\frac{k_1-1}{n_1},\frac{k_1}{n_1},\frac{k_1+1}{n_1};|\alpha-x|\right]_\alpha,\ u_{n_1,k_1} \in C[0,1],$$

with a similar definition holding for $u_{n_2,k_2}(y)$.

The bivariate operator $S_{\Delta_{n_1},\Delta_{n_2}}$ can also be defined by

$$S_{\Delta_{n_1},\Delta_{n_2}} f(x,y) := \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right)\cdot u_{n_1,k_1}(x)\cdot u_{n_2,k_2}(y),$$

for $f \in C(I)$, $x, y \in X$, $u_{n_i,k_i} \in C[0,1]$, $i = 1,2$,

$$u_{n_i,k_i}\left(\frac{l_i}{n_i}\right) = \delta_{k_i,l_i},\ k_i, l_i = 0,\dots,n_i,\ i = 1,2.$$

We now give the second moment of this operator in the bivariate case.

For $x \in \left[\frac{k_1-1}{n_1},\frac{k_1}{n_1}\right]$, $y = \left[\frac{k_2-1}{n_2},\frac{k_2}{n_2}\right]$, we get

$$\begin{aligned}
S_{\Delta_1,\Delta_2}\left(d_2^2(\cdot,(x,y);(x,y))\right) &= \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} u_{n_1,i_1}(x)u_{n_2,i_2}(y)d_2^2\left(\left(\frac{i_1}{n_1},\frac{i_2}{n_2}\right),(x,y)\right)\\
&= \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} u_{n_1,i_1}(x)u_{n_2,i_2}(y)\left\{\left(\frac{i_1}{n_1}-x\right)^2+\left(\frac{i_2}{n_2}-y\right)^2\right\}\\
&= \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} u_{n_1,i_1}(x)u_{n_2,i_2}(y)\left(\frac{i_1}{n_1}-x\right)^2+\sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} u_{n_1,i_1}(x)u_{n_2,i_2}(y)\left(\frac{i_2}{n_2}-y\right)^2\\
&= \sum_{i_1=0}^{n_1} u_{n_1,i_1}(x)\left(\frac{i_1}{n_1}-x\right)^2+\sum_{i_2=0}^{n_2} u_{n_2,i_2}(y)\left(\frac{i_2}{n_2}-y\right)^2\\
&= \sum_{i_1=0}^{n_1}\frac{n_1}{2}\left(\frac{i_1}{n_1}-x\right)^2\left\{\left|\frac{i_1+1}{n_1}-x\right|-2\left|\frac{i_1}{n_1}-x\right|+\left|\frac{i_1-1}{n_1}-x\right|\right\}\\
&+ \sum_{i_2=0}^{n_2}\frac{n_2}{2}\left(\frac{i_2}{n_2}-y\right)^2\left\{\left|\frac{i_2+1}{n_2}-y\right|-2\left|\frac{i_2}{n_2}-y\right|+\left|\frac{i_2-1}{n_2}-y\right|\right\}\\
&= \left(x-\frac{k_1-1}{n_1}\right)\left(\frac{k_1}{n_1}-x\right)+\left(y-\frac{k_2-1}{n_2}\right)\left(\frac{k_2}{n_2}-y\right).
\end{aligned}$$

This quantity attains its maximum when both $x = \frac{2k_1-1}{2n_1}$ and $y = \frac{2k_2-1}{2n_2}$, which implies

$$S_{\Delta_{n_1},\Delta_{n_2}}(d_2^2(\cdot,(x,y));(x,y)) \leq \frac{1}{4n_1^2} + \frac{1}{4n_2^2}.$$

3.2.9 Bivariate BLaC operators

We define the bivariate BLaC operator and then derive a Chebyshev-Grüss inequality.

Let $X = [0,1]$ and $I = [0,1] \times [0,1]$ be the compact metric space equipped with the Euclidean metric

$$d_2((s,t),(x,y)) := \sqrt{(s-x)^2 + (t-y)^2},$$

for $(s,t),(x,y) \in I$. The two-dimensional scaling functions $\varphi_\Delta(x,y)$ are given by

$$\varphi_\Delta(x,y) = \varphi_\Delta(x) \cdot \varphi_\Delta(y) := \begin{cases} \varphi_\Delta(x), & \text{for } \Delta \leq y < 1, \\ \varphi_\Delta(y), & \text{for } \Delta \leq x < 1, \\ \frac{xy}{\Delta^2}, & \text{for } 0 \leq x,y < \Delta, \\ -\frac{x}{\Delta^2} \cdot (y-1-\Delta), & \text{for } 0 \leq x < \Delta,\ 1 \leq y < 1+\Delta, \\ -\frac{y}{\Delta^2} \cdot (x-1-\Delta), & \text{for } 0 \leq y < \Delta,\ 1 \leq x < 1+\Delta, \\ \frac{1}{\Delta^2} \cdot (x-1-\Delta)(y-1-\Delta), & \text{for } 1 \leq x,y < 1+\Delta, \\ 0, & \text{else}, \end{cases}$$

and the bivariate fundamental functions $\varphi_{i,j}^n(x,y)$ are defined by:

$$\varphi_{i,j}^n(x,y) = \varphi_\Delta(2^n(x - i2^{-n})) \cdot \varphi_\Delta(2^n(y - j2^{-n})).$$

The interpolation points $\eta_{i,j}^n \in \mathbb{R}^2$ look as follows:

$$\begin{aligned} \eta_{i,j}^n &= \left(\frac{2i+1+\Delta}{2^{n+1}}, \frac{2j+1+\Delta}{2^{n+1}} \right), \text{ for } i,j = 0,\dots,2^n-2, \\ \eta_{-1,-1}^n &= (0,0),\ \eta_{2^n-1,2^n-1}^n = (1,1), \\ \eta_{-1,j}^n &= \left(0, \frac{2j+1+\Delta}{2^{n+1}}\right),\ \eta_{i,-1}^n = \left(\frac{2i+1+\Delta}{2^{n+1}}, 0\right), \\ \eta_{2^n-1,j}^n &= \left(1, \frac{2j+1+\Delta}{2^{n+1}}\right),\ \eta_{i,2^n-1}^n = \left(\frac{2i+1+\Delta}{2^{n+1}}, 1\right). \end{aligned}$$

Definition 3.2.8. (see [87]) For $f \in C(I)$ and $x,y \in X$ the *bivariate BLaC operator* is given by

$$BL_n f(x,y) = BL_n f(x,y,n) := \sum_{i=-1}^{2^n-1} \sum_{j=-1}^{2^n-1} f(\eta_{i,j}^n) \cdot \varphi_{i,j}^n(x,y).$$

Remark 3.2.9. The properties of the BLaC operator in the univariate case also apply here.

The second moment of the operator will be described in the proof of Theorem 3.3.13, in one of the next sections, for the different cases.

3.2.10 Products of Mirakjan-Favard-Szász operators

In [116], a bivariate Mirakjan-Favard-Szász operator is considered, using the approach of parametric extensions.

Take $X = [0,\infty)$ and $\mathbb{R}^X$ the space of real-valued functions defined on X. The Mirakjan-Favard-Szász operator is defined as before. If we consider a bivariate function defined on $[0,\infty) \times [0,\infty) = I$, then the parametric extensions of the operator $M_n f$ are given by

$$({}_xM_{n_1}f)(x,y) := e^{-n_1x} \sum_{k_1=0}^{\infty} \frac{(n_1x)^{k_1}}{k_1!} f\left(\frac{k_1}{n_1}, y\right),$$
$$({}_yM_{n_2}f)(x,y) := e^{-n_2y} \sum_{k_2=0}^{\infty} \frac{(n_2y)^{k_2}}{k_2!} f\left(x, \frac{k_2}{n_2}\right).$$

Since we know that M_n is a positive linear operator, we have that its parametric extensions ${}_xM_{n_1}$ and ${}_yM_{n_2}$ are also positive linear operators.

Proposition 3.2.10. *The parametric extensions ${}_xM_{n_1}$, ${}_yM_{n_2}$ satisfy the relation*

$${}_xM_{n_1} \cdot_y M_{n_2} =_y M_{n_2} \cdot_x M_{n_1}.$$

Their product is the bidimensional operator M_{n_1,n_2}, which, for every function $f \in \mathbb{R}^I$ looks as:

$$M_{n_1,n_2}(f;(x,y)) := e^{-n_1x} \cdot e^{-n_2y} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(n_1x)^{k_1}}{k_1!} \cdot \frac{(n_2y)^{k_2}}{k_2!} \cdot f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}\right).$$

J. Favard was the first to introduce these bivariate operators (see [37]).

3.2.11 Products of Baskakov operators

In [61], Baskakov operators for functions of two variables are studied.

We consider $X = [0,\infty)$ and $\mathbb{R}^X$ the space of real-valued functions defined on X. The Baskakov operator is defined as before. If we consider a bivariate function defined on $[0,\infty) \times [0,\infty) = I$, then the parametric extensions of the operator $A_n f$ are given by

$$({}_xA_{n_1}f)(x,y) := \sum_{k_1=0}^{\infty} \binom{n_1+k_1-1}{k_1} \frac{x^{k_1}}{(1+x)^{n_1+k_1}} f\left(\frac{k_1}{n_1}, y\right),$$
$$({}_yA_{n_2}f)(x,y) := \sum_{k_2=0}^{\infty} \binom{n_2+k_2-1}{k_2} \frac{y^{k_2}}{(1+y)^{n_2+k_2}} f\left(x, \frac{k_2}{n_2}\right).$$

We know that A_n is a positive linear operator, we have that its parametric extensions ${}_xA_{n_1}$ and ${}_yA_{n_2}$ are also positive linear operators.

Proposition 3.2.11. *The parametric extensions ${}_xA_{n_1}$, ${}_yA_{n_2}$ satisfy the relation*

$${}_xA_{n_1} \cdot_y A_{n_2} =_y A_{n_2} \cdot_x A_{n_1}.$$

Their product is the bidimensional operator A_{n_1,n_2}, which, for every function $f \in \mathbb{R}^I$, looks as:

$$\begin{aligned} &A_{n_1,n_2}(f;(x,y)) \\ &:= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\binom{n_1+k_1-1}{k_1}\binom{n_2+k_2-1}{k_2}\frac{x^{k_1}}{(1+x)^{n_1+k_1}}\frac{y^{k_2}}{(1+y)^{n_2+k_2}}f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right) \\ &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} a_{n_1,k_1}(x)\cdot a_{n_2,k_2}(y)\cdot f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right), \end{aligned}$$

for $(x,y) \in I$, $n_1, n_2 \in \mathbb{N}$.

3.2.12 Products of Lagrange operators

A bivariate Lagrange interpolation operator is considered (see [39]), using the approach of parametric extensions.

Consider $X = [-1,1]$ and $\mathbb{R}^X$ the space of real-valued functions defined on X. The univariate Lagrange operator is defined as before. If we consider a bivariate function defined on $[-1,1]\times[-1,1] = I$, then the parametric extensions of the operator $L_n f$ are given by

$$({}_xL_{n_1}f)(x,y) := \sum_{k_1=1}^{n_1} l_{k_1,n_1}(x)\cdot f(x_{k_1,n_1},y),$$

$$({}_yL_{n_2}f)(x,y) := \sum_{k_2=1}^{n_2} l_{k_2,n_2}(y)\cdot f(x,y_{k_2,n_2}).$$

Since we know that L_n is a linear operator (only in exceptional cases positive), its parametric extensions ${}_xL_{n_1}$ and ${}_yL_{n_2}$ are also linear operators that are only sometimes positive.

Proposition 3.2.12. *The parametric extensions ${}_xL_{n_1}$, ${}_yL_{n_2}$ satisfy the relation*

$${}_xL_{n_1}\cdot_y L_{n_2} =_y L_{n_2}\cdot_x L_{n_1}.$$

Their product is the bidimensional operator L_{n_1,n_2}, which, for every function $f \in \mathbb{R}^I$ looks as:

$$L_{n_1,n_2}(f;(x,y)) := \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} f(x_{k_1,n_1},y_{k_2,n_2})\cdot l_{k_1,n_1}(x)\cdot l_{k_2,n_2}(y).$$

The Lagrange fundamental functions are given as usual

$$l_{k_1,n_1}(x) = \frac{w_{n_1}(x)}{w'_{n_1}(x_{k_1,n_1})(x-x_{k_1,n_1})},\ 1 \le k_1 \le n_1,$$

where $w_{n_1}(x) = \prod_{k_1=1}^{n_1}(x - x_{k_1,n_1})$. The fundamental functions $l_{k_2,n_2}(y)$ are defined analogously.

The corresponding Lebesgue functions are

$$\Lambda_{n_1}(x) := \sum_{k_1=1}^{n_1}|l_{k_1,n_1}|$$

and

$$\Lambda_{n_2}(y) := \sum_{k_2=1}^{n_2} |l_{k_2,n_2}|.$$

Regarding the sums of the squared fundamental functions of a Lagrange interpolation based upon any infinite matrix X, we recall a result from [64]. It holds, for $\alpha = 2$ in the relation (3.1) in the cited article, that

$$\sum_{k_1=1}^{n_1} l_{k_1,n_1}^2(x) \geq \frac{1}{4}, \text{ for } -1 \leq x \leq 1,$$

and the same holds for the squares of the fundamental functions with respect to y.

3.3 Main Results

3.3.1 (Pre-)Chebyshev-Grüss inequalities in the bivariate case

We have the following result for the bivariate case, result that can be directly obtained from Theorem 2.2.36.

Theorem 3.3.1. *Let (X, d) be a compact metric space and denote $I := X \times X$. If $f, g \in C(I)$ and $x, y \in X$ fixed, then the inequality*

$$\begin{aligned}
&|T(f,g;(x,y))| \\
&\leq \frac{1}{4}\tilde{\omega}_d\left(f;4\sqrt{H(d^2(\cdot,(x,y));(x,y))}\right)\cdot\tilde{\omega}_d\left(g;4\sqrt{H(d^2(\cdot,(x,y));(x,y))}\right) \\
&= \frac{1}{4}\tilde{\omega}_d\left(f;4\sqrt{H(d_X^2(\cdot,x);x)+H(d_X^2(\cdot,y);y)}\right)\cdot\tilde{\omega}_d\left(g;4\sqrt{H(d_X^2(\cdot,x);x)+H(d_X^2(\cdot,y);y)}\right)
\end{aligned}$$

holds, where $H(d^2(\cdot,(x,y));(x,y))$ is the second moment of the bivariate operator H. We consider here the Euclidean metric d_2.

A pre-Chebyshev-Grüss inequality, that directly follows from Theorem 2.2.37, can also be given.

Theorem 3.3.2. *If $f, g \in C(I)$, where (X, d) is a compact metric space and $I = X \times X$, and $x, y \in X$ fixed, then the inequality*

$$|T(f,g;(x,y))| \leq \frac{1}{2}\cdot\min\{A,B\}$$

holds, where

$$\begin{aligned}
A &:= \|f\|_\infty\cdot\tilde{\omega}_d(g;4\cdot H(d(\cdot,(x,y));(x,y))) \\
B &:= \|g\|_\infty\cdot\tilde{\omega}_d(f;4\cdot H(d(\cdot,(x,y));(x,y))),
\end{aligned}$$

and the metric that we need is $d_1((s,t),(x,y)) := |s-x|+|t-y|$.

3.3.2 Applications to bivariate positive linear operators

3.3.2.1 The bivariate Bernstein operator

The Chebyshev-Grüss inequality involving second moments for the tensor product Bernstein operator looks as follows.

Theorem 3.3.3. *If we take $H = B_{n_1,n_2}$ in Theorem 3.3.1, we get*

$$\begin{aligned}
&|B_{n_1,n_2}(f \cdot g;(x,y)) - B_{n_1,n_2}(f;(x,y)) \cdot B_{n_1,n_2}(g;(x,y))| \\
&\le \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;4\cdot\sqrt{\frac{x(1-x)}{n_1}+\frac{y(1-y)}{n_2}}\right)\cdot\widetilde{\omega}_{d_2}\left(g;4\cdot\sqrt{\frac{x(1-x)}{n_1}+\frac{y(1-y)}{n_2}}\right),
\end{aligned} \tag{3.3.1}$$

which implies

$$\begin{aligned}
|T(f,g;(x,y))| &\le \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;4\sqrt{\frac{1}{4n_1}+\frac{1}{4n_2}}\right)\cdot\widetilde{\omega}_{d_2}\left(g;4\sqrt{\frac{1}{4n_1}+\frac{1}{4n_2}}\right) \\
&= \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;2\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\right)\cdot\widetilde{\omega}_{d_2}\left(g;2\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\right),
\end{aligned} \tag{3.3.2}$$

for two functions $f,g \in C(I)$, $I = [0,1]\times[0,1]$ and $x,y \in [0,1]$ fixed.

3.3.2.2 Bivariate special King operators

Theorem 3.3.4. *If we take $H = V^*_{n_1,n_2}$ in Theorem 3.3.1 and consider the second moments of these operators for the case $n_1 = n_2 = 2,3,\ldots$, $n_1 \neq n_2$, we obtain the following inequality.*

$$\begin{aligned}
&|T(f,g;(x_1,x_2))| \\
&\le \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;4\sqrt{\sum_{i=1}^{2}2x_i(x_i-r^*_{n_i}(x_i))}\right)\cdot\widetilde{\omega}_{d_2}\left(g;4\sqrt{\sum_{i=1}^{2}2x_i(x_i-r^*_{n_i}(x_i))}\right) \\
&= \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;4\sqrt{\sum_{i=1}^{2}2x_i\left(x_i+\frac{1}{2(n_i-1)}-\sqrt{\frac{n_i}{n_i-1}x_i^2+\frac{1}{4(n_i-1)^2}}\right)}\right) \\
&\cdot\widetilde{\omega}_{d_2}\left(g;4\sqrt{\sum_{i=1}^{2}2x_i\left(x_i+\frac{1}{2(n_i-1)}-\sqrt{\frac{n_i}{n_i-1}x_i^2+\frac{1}{4(n_i-1)^2}}\right)}\right).
\end{aligned}$$

For the other cases, as well as for the general bivariate operator V_{n_1,n_2} and for $V^{min}_{n_1,n_2}$, results can be obtained in a similar way.

3.3.2.3 Products of Hermite-Fejér operators

The Chebyshev-Grüss inequality involving second moments for the tensor product Hermite-Fejér operator is given in the sequel.

Theorem 3.3.5. *If we take $H = H_{2n_1-1,2n_2-1}$ in Theorem 3.3.1, we get*

$$
\begin{aligned}
&|H_{2n_1-1,2n_2-1}(f \cdot g;(x,y)) - H_{2n_1-1,2n_2-1}(f;(x,y)) \cdot H_{2n_1-1,2n_2-1}(g;(x,y))| \\
&\leq \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;4\sqrt{\frac{1}{n_1}T_{n_1}^2(x)+\frac{1}{n_2}T_{n_2}^2(y)}\right)\cdot\widetilde{\omega}_{d_2}\left(g;4\sqrt{\frac{1}{n_1}T_{n_1}^2(x)+\frac{1}{n_2}T_{n_2}^2(y)}\right)
\end{aligned}
$$

for two functions $f, g \in C(I)$, $I = [-1,1] \times [-1,1]$ and $x_1, x_2 \in [-1,1]$ fixed.

Theorem 3.3.6. *If we replace $H = H_{2n_1-1,2n_2-1}$ in Theorem 3.3.2, the pre-Chebyshev-Grüss-type inequality looks as follows:*

$$
\begin{aligned}
|T(f,g;(x,y))| &\leq \frac{1}{2}\min\left\{\|f\|_\infty \cdot \widetilde{\omega}_{d_1}(g;4\cdot D);\|g\|_\infty \cdot \widetilde{\omega}_{d_1}(f;4\cdot D)\right\} \\
&\leq \frac{1}{2}\min\left\{\|f\|_\infty \cdot \widetilde{\omega}_{d_1}(f;40\cdot E);\|g\|_\infty \cdot \widetilde{\omega}_{d_1}(g;40\cdot E)\right\},
\end{aligned}
$$

where $D := \frac{4}{n_1}\cdot|T_{n_1}(x)|\cdot\{\sqrt{1-x^2}\cdot\ln n_1+1\}+\frac{4}{n_2}\cdot|T_{n_2}(y)|\cdot\{\sqrt{1-y^2}\cdot\ln n_2+1\}$ and $E := \frac{|T_{n_1}(x)|\ln n_1}{n_1}+\frac{|T_{n_2}(y)|\ln n_2}{n_2}$.

3.3.2.4 Products of quasi-Hermite-Fejér operators

The Chebyshev-Grüss inequality involving second moments for the tensor product quasi-Hermite-Fejér operator is given in the sequel.

Theorem 3.3.7. *If we take $H = Q_{n_1,n_2}$ in Theorem 3.3.1, we get*

$$
\begin{aligned}
&|Q_{n_1,n_2}(f \cdot g;(x,y)) - Q_{n_1,n_2}(f;(x,y)) \cdot Q_{n_1,n_2}(g;(x,y))| \\
&\leq \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;4\sqrt{\left(\frac{1}{n_1+1}(1-x^2)\cdot U_{n_1}^2(x)+\frac{1}{n_2+1}(1-y^2)\cdot U_{n_2}^2(y)\right)}\right) \\
&\cdot\widetilde{\omega}_{d_2}\left(g;4\sqrt{\left(\frac{1}{n_1+1}(1-x^2)\cdot U_{n_1}^2(x)+\frac{1}{n_2+1}(1-y^2)\cdot U_{n_2}^2(y)\right)}\right)
\end{aligned}
$$

for two functions $f, g \in C(I)$, $I = [-1,1] \times [-1,1]$ and $x, y \in [-1,1]$ fixed.

Theorem 3.3.8. *A pre-Chebyshev-Grüss inequality in the bivariate case looks as follows.*

$$
|T(f,g;(x,y))| \leq \frac{1}{2}\min\{\|f\|_\infty \cdot \widetilde{\omega}_{d_1}(g;40\cdot F);\|g\|_\infty \cdot \widetilde{\omega}_{d_1}(f;40\cdot F)\},
$$

where $F := (1-x^2)\cdot\frac{1}{n_1+1}\cdot|U_{n_1}(x)|\cdot\ln(n_1+1)+(1-y^2)\cdot\frac{1}{n_2+1}\cdot|U_{n_2}(y)|\cdot\ln(n_2+1)$.

3.3.2.5 Products of almost-Hermite-Fejér operators

The Chebyshev-Grüss inequality involving second moments for the tensor product almost-Hermite-Fejér operator is given in the sequel.

Theorem 3.3.9. *If we take $H = F^{(\frac{1}{2},-\frac{1}{2})}_{1,0;n_1,n_2}$ in Theorem 3.3.1, we get*

$$\left| F^{(\frac{1}{2},-\frac{1}{2})}_{1,0;n_1,n_2}(f \cdot g;(x,y)) - F^{(\frac{1}{2},-\frac{1}{2})}_{1,0;n_1,n_2}(f;(x,y)) \cdot F^{(\frac{1}{2},-\frac{1}{2})}_{1,0;n_1,n_2}(g;(x,y)) \right|$$
$$\leq \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;\frac{4\sqrt{2}}{\sqrt{3}}\sqrt{\frac{(1-x)w(x)^2}{n_1}+\frac{(1-y)w(y)^2}{n_2}}\right)$$
$$\cdot\, \widetilde{\omega}_{d_2}\left(g;\frac{4\sqrt{2}}{\sqrt{3}}\sqrt{\frac{(1-x)w(x)^2}{n_1}+\frac{(1-y)w(y)^2}{n_2}}\right)$$

for two functions $f,g \in C([-1,1]^2)$ and $x,y \in [-1,1]$ fixed.

Theorem 3.3.10. *A pre-Chebyshev-Grüss inequality in the bivariate case looks as follows.*

$$|T(f,g;(x,y))| \leq \frac{1}{2}\min\{\|f\|_\infty \cdot \widetilde{\omega}_{d_1}(g;4\cdot G); \|g\|_\infty \cdot \widetilde{\omega}_{d_1}(f;4\cdot G)\},$$

where $G := c_1 \cdot \frac{\sqrt{1-x^2}\cdot \ln n_1 + 1}{2n_1+1} + c_2 \cdot \frac{\sqrt{1-y^2}\cdot \ln n_2+1}{2n_2+1}$.

3.3.2.6 Bivariate Shepard-type operators

Some pre-Chebyshev-Grüss inequalities for the bivariate CBS operator based on pairs of equidistant points are given as follows.

Theorem 3.3.11. *Let $I = [0,1]^2$, $f,g \in C(I)$ and $x,y \in [0,1]$ fixed. Then if we take $H = S^{\mu_1,\mu_2}_{n_1+1,n_2+1}$ in Theorem 3.3.2, then the inequality*

$$|T(f,g;(x,y))| \leq \frac{1}{2}\min\{A,B\}$$

holds, where

$$A := \|f\|_\infty \cdot \widetilde{\omega}_{d_1}\left(g;4S^{\mu_1,\mu_2}_{n_1+1,n_2+1}(d_1(\cdot,(x,y));(x,y))\right)$$
$$B := \|g\|_\infty \cdot \widetilde{\omega}_{d_1}\left(f;4S^{\mu_1,\mu_1}_{n_1+1,n_2+1}(d_1(\cdot,(x,y));(x,y))\right)$$

Proof. The first absolute moment of the bivariate CBS operator is given by:

$$S^{\mu_1,\mu_2}_{n_1+1,n_2+1}(d_1(\cdot,(x,y);(x,y)) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} s_{k_1,\mu_1}(x)\cdot s_{k_2,\mu_2}(y)\cdot\left[\left|\frac{k_1}{n_1}-x\right|+\left|\frac{k_2}{n_2}-y\right|\right]$$

$$= \sum_{k_1=0}^{n_1} s_{k_1,\mu_1}(x)\left(\sum_{k_2=0}^{n_2} s_{k_2,\mu_2}(y)\cdot\left[\left|\frac{k_1}{n_1}-x\right|+\left|\frac{k_2}{n_2}-y\right|\right]\right)$$

$$= \sum_{k_1=0}^{n_1} s_{k_1,\mu_1}(x)\left[\left|\frac{k_1}{n_1}-x\right|\cdot\underbrace{\sum_{k_2=0}^{n_2} s_{k_2,\mu_2}(y)}_{=1}+\sum_{k_2=0}^{n_2} s_{k_2,\mu_2}(y)\cdot\left|\frac{k_2}{n_2}-y\right|\right]$$

$$= \sum_{k_1=0}^{n_1} s_{k_1,\mu_1}(x)\left[\left|\frac{k_1}{n_1}-x\right|+\sum_{k_2=0}^{n_2}\frac{\left|y-\frac{k_2}{n_2}\right|^{-\mu_2}}{\sum_{l=0,l\neq k_2}^{n_2}\left|y-\frac{l}{n_2}\right|^{-\mu_2}}\cdot\left|\frac{k_2}{n_2}-y\right|\right]$$

$$= \sum_{k_1=0}^{n_1} s_{k_1,\mu_1}(x)\left[\left|\frac{k_1}{n_1}-x\right|+\underbrace{\sum_{k_2=0}^{n_2}\frac{\left|y-\frac{k_2}{n_2}\right|^{1-\mu_2}}{\sum_{l=0}^{n_2}\left|y-\frac{l}{n_2}\right|^{-\mu_2}}}_{=S^{\mu_2}_{n_2+1}(d(\cdot,y);y)}\right]$$

$$= S^{\mu_1}_{n_1+1}(|\cdot-x|\,;x)+S^{\mu_2}_{n_2+1}(|\cdot-y|\,;y)$$

We need to discriminate between some cases for μ_1 and μ_2, so we will obtain different inequalities in each of the cases.

In the first case, for $\mu_1=\mu_2=1$, we have

$$S^{1,1}_{n_1+1,n_2+1}(d_1(\cdot,(x,y));(x,y)) = \sum_{k_1=0}^{n_1}\frac{1}{\sum\limits_{l=0}^{n_1}\left|x-\frac{l}{n_1}\right|^{-1}}+\sum_{k_2=0}^{n_2}\frac{1}{\sum\limits_{l=0}^{n_2}\left|y-\frac{l}{n_2}\right|^{-1}}$$

$$= (n_1+1)\left(\sum_{l=0}^{n_1}\frac{1}{\left|x-\frac{l}{n_1}\right|}\right)^{-1}+(n_2+1)\left(\sum_{l=0}^{n_2}\frac{1}{\left|y-\frac{l}{n_2}\right|}\right)^{-1}$$

Let l_0 be defined by $\frac{l_0}{n_1}<x<\frac{l_0+1}{n_1}$. Then we obtain

$$\begin{aligned}\frac{1}{n_1+1}\left(\sum_{l=0}^{n_1}\frac{1}{\left|x-\frac{l}{n_1}\right|}\right) &\geq \frac{n_1}{n_1+1}\left\{\sum_{l=0}^{l_0}\frac{1}{l_0+1-l}+\sum_{l=l_0+1}^{n_1}\frac{1}{l-l_0}\right\}\\ &\geq \frac{n_1}{n_1+1}\left\{\int_1^{l_0+2}\frac{1}{x}\mathrm{dx}+\int_1^{n_1-l_0+1}\frac{1}{x}\mathrm{dx}\right\}\\ &= \frac{n_1}{n_1+1}\ln((l_0+2)\cdot(n_1-l_0+1))\\ &\geq \frac{n_1}{n_1+1}\ln(2n_1+2), n_1\geq 1+l_0,\end{aligned}$$

so it holds

$$S^1_{n_1+1}(d_1(\cdot,x);x) \leq \frac{n_1+1}{n_1\ln(2n_1+2)}.$$

Analogue, we have

$$S^1_{n_2+1}(d_1(\cdot,y);y) \leq \frac{n_2+1}{n_2\ln(2n_2+2)}$$

and taking both results together we get

$$S^{1,1}_{n_1+1,n_2+1}(d_1(\cdot,(x,y));(x,y)) \leq \frac{n_1+1}{n_1\ln(2n_1+2)} + \frac{n_2+1}{n_2\ln(2n_2+2)}.$$

For the next two cases, for $1<\mu_1<2$ and $1<\mu_2<2$ and $\mu_1=\mu_2=2$ we consider l_0 defined by

$$\left|x-\frac{l_0}{n_1}\right| = \min\left\{\left|x-\frac{l}{n_1}\right| : 0\leq l\leq n_1\right\}.$$

Then we have

$$\begin{aligned}
S^{\mu_1}_{n_1+1}(d_1(\cdot,x);x) &\leq |x-x_{l_0}|^{\mu_1}\cdot\sum_{k_1=0}^{n_1}|x-x_{k_1}|^{1-\mu_1} \\
&\leq \frac{1}{n_1}+\left(\frac{1}{n_1}\right)\cdot\left\{\sum_{k_1<l_0}|x-x_{k_1}|^{1-\mu_1}+\sum_{k_1>l_0}|x-x_{k_1}|^{1-\mu_1}\right\} \\
&\leq \frac{1}{n_1}+\left(\frac{1}{n_1}\right)\cdot\left\{\sum_{l=0}^{l_0-1}\left(\frac{1}{2}+l\right)^{1-\mu_1}+\sum_{l=0}^{n_1-l_0-1}\left(\frac{1}{2}+l\right)^{1-\mu_1}\right\},
\end{aligned}$$

and after some calculations we obtain

$$S^{\mu_1}_{n_1+1}(d_1(\cdot,x);x) \leq \begin{cases} \frac{1}{n_1}+\frac{1}{n_1}\left[2^{\mu_1}+\frac{2}{2-\mu_1}\left(\frac{n_1+1}{2}\right)^{2-\mu_1}\right] & \text{, for } 1<\mu_1<2 \\ \frac{5+2\ln(n_1+1)}{n_1} & \text{, for } \mu_1=2 \end{cases}$$

and

$$S^{\mu_2}_{n_2+1}(d_1(\cdot,y);y) \leq \begin{cases} \frac{1}{n_2}+\frac{1}{n_2}\left[2^{\mu_2}+\frac{2}{2-\mu_2}\left(\frac{n_2+1}{2}\right)^{2-\mu_2}\right] & \text{, for } 1<\mu_2<2 \\ \frac{5+2\ln(n_2+1)}{n_2} & \text{, for } \mu_2=2 \end{cases},$$

respectively.

In the case $\mu_1,\mu_2>2$ we can also give the bivariate pre-Chebyshev-Grüss inequality. The first absolute moment for the bivariate CBS operator is given by

$$\begin{aligned}
S^{\mu_1,\mu_2}_{n_1+1,n_2+1}(d_1(\cdot,(x,y));(x,y)) &= S^{\mu_1}_{n_1+1}(d_1(\cdot,x);x)+S^{\mu_2}_{n_2+1}(d_1(\cdot,y);y) \\
&\leq \frac{3}{n_1}+\frac{3}{n_2} = 3\left(\frac{1}{n_1}+\frac{1}{n_2}\right),
\end{aligned}$$

and using this we get a bivariate Chebyshev-Grüss inequality. □

Let us now consider the bivariate CBS operator $S^{\mu_1,\mu_2}_{n_1,n_2}$ defined on $I=[0,1]\times[0,1]$ and take the metric

$$d_2((s,t),(x,y)) := \sqrt{(s-x)^2+(t-y)^2}.$$

Now take this special case of the CBS operator and apply Theorem 3.3.1 to it. Then we get

Theorem 3.3.12. *If $f, g \in C(I)$, then the inequality*

$$\begin{aligned}&|T(f,g;(x,y))|\\&\leq \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;4\sqrt{S^{\mu_1,\mu_2}_{n_1,n_2}(d_2^2(\cdot,(x,y));(x,y))}\right)\cdot\widetilde{\omega}_{d_2}\left(g;4\sqrt{S^{\mu_1,\mu_2}_{n_1,n_2}(d_2^2(\cdot,(x,y));(x,y))}\right)\end{aligned}$$

holds.

Proof. Just like in the proof of the pre-Chebyshev-Grüss inequality for the bivariate CBS operator with equidistant points, one can prove that

$$S^{\mu_1,\mu_2}_{n_1,n_2}(d_2^2(\cdot,(x,y));(x,y)) = S^{\mu_1}_{n_1}(d_2^2(\cdot,x);x) + S^{\mu_2}_{n_2}(d_2^2(\cdot,y);y)$$

is true. If we discriminate between different cases, for $\mu_1 = \mu_2 = 1$, $1 < \mu_1, \mu_2 < 2$ and $\mu_1 \geq \mu_2$, and the last case $\mu_1, \mu_2 > 2$, we obtain again additional inequalities. The details are similar to the ones for the pre-Chebyshev-Grüss inequalities. □

3.3.2.7 Tensor product BLaC operator

We apply Theorem 3.3.1 to the bivariate BLaC operator and obtain:

Theorem 3.3.13. *If $f, g \in C(I)$, where $I = [0,1]^2$, then the inequality*

$$\begin{aligned}|T(f,g;(x,y))| &\leq \frac{1}{4}\widetilde{\omega}_{d_2}(f;4\sqrt{BL_n(d_2^2(\cdot,(x,y));(x,y))})\cdot\widetilde{\omega}_{d_2}(g;4\sqrt{BL_n(d_2^2(\cdot,(x,y));(x,y))})\\&\leq \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;\frac{\sqrt{2}}{2^{n-4}}\right)\cdot\widetilde{\omega}_{d_2}\left(g;\frac{\sqrt{2}}{2^{n-4}}\right)\end{aligned}$$

holds.

Proof. The second moment of the bivariate BLaC operator is of interest here.

Let $x \in [\frac{k}{2^n}, \frac{k+1}{2^n}]$, $y \in [\frac{l}{2^n}, \frac{l+1}{2^n}]$, $k, l \in \{0, \ldots, 2^n - 1\}$. For $x = 1$ and $y = 1$, take $k = 2^n - 1$ and $l = 2^n - 1$, respectively. The second moment of the bidimensional BLaC operator is:

$$\begin{aligned}BL_n(d_2^2(\cdot,(x,y));(x,y)) &= \underbrace{\sum_{i=-1}^{2^n-1}\sum_{j=-1}^{2^n-1} d_2^2(\eta^n_{i,j},(x,y))\cdot\varphi^n_{i,j}(x,y)}_{(1)}\\&= (\underbrace{d_2^2(\eta^n_{k-1,l-1},(x,y))}_{\leq\frac{1}{2^{2n-3}}}\underbrace{\varphi^n_{k-1,l-1}(x,y)}_{\leq 1} + \underbrace{d_2^2(\eta^n_{k,l-1},(x,y))\varphi^n_{k,l-1}(x,y)}_{\leq\frac{1}{2^{2n-3}}\cdot 1}\\&+ \underbrace{d_2^2(\eta^n_{k-1,l},(x,y))\varphi^n_{k-1,l}(x,y)}_{\leq\frac{1}{2^{2n-3}}\cdot 1} + \underbrace{d_2^2(\eta^n_{k,l},(x,y))\varphi^n_{k,l}(x,y)}_{\leq\frac{1}{2^{2n-3}}\cdot 1})\\&\leq \frac{4}{2^{2n-3}} = \frac{1}{2^{2n-5}} = \frac{2}{2^{2(n-2)}},\end{aligned}$$

so the sum (1) has at most 4 terms. The idea of the above calculations is that the maximum distance in each component is smaller than or equal to $\frac{1}{2^{n-1}}$, so we have:

$$d_2^2(\eta^n_{k-1,l-1},(x,y)) \leq \left(\frac{1}{2^{n-1}}\right)^2 + \left(\frac{1}{2^{n-1}}\right)^2 = \frac{2}{2^{2n-2}} = \frac{1}{2^{2n-3}}.$$

Then we get

$$\begin{aligned} BL_n(d_2^2(\cdot,(x,y));(x,y))^{\frac{1}{2}} &= \sqrt{BL_n(d_2^2(\cdot,(x,y));(x,y))} \\ &= \sqrt{\sum_{i=-1}^{2^n-1}\sum_{j=-1}^{2^n-1} d_2^2(\eta_{i,j}^n,(x,y))\cdot\varphi_{i,j}^n(x,y)} \\ &\le \sqrt{4\left(\frac{1}{2^{2n-3}}\right)} = \sqrt{\frac{2}{2^{2(n-2)}}}. \end{aligned}$$

The Chebyshev-Grüss inequality becomes:

$$\begin{aligned} |T(f,g;(x,y))| &\le \frac{1}{4}\widetilde{\omega}_{d_2}(f;4\sqrt{BL_n(d_2^2(\cdot,(x,y));(x,y))})\cdot\widetilde{\omega}_{d_2}(g;4\sqrt{BL_n(d_2^2(\cdot,(x,y));(x,y))}) \\ &\le \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;4\sqrt{\frac{4}{2^{2n-3}}}\right)\cdot\widetilde{\omega}_{d_2}\left(g;4\sqrt{\frac{4}{2^{2n-3}}}\right) \\ &= \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;4\sqrt{\frac{1}{2^{2n-5}}}\right)\cdot\widetilde{\omega}_{d_2}\left(g;4\sqrt{\frac{1}{2^{2n-5}}}\right) \\ &= \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;\frac{\sqrt{2}}{2^{n-4}}\right)\cdot\widetilde{\omega}_{d_2}\left(g;\frac{\sqrt{2}}{2^{n-4}}\right), \end{aligned}$$

for $f,g\in C(I)$, where $I=[0,1]^2$. Our proof is completed. □

The bivariate case is of particular interest because it can be applied in the image compression process (for examples, see [16] and [87]).

3.3.2.8 Bivariate piecewise linear interpolation operator at equidistant knots

We consider $H = S_{\Delta_{n_1},\Delta_{n_2}}$ in Theorem 3.3.1 and, using the second moment of this bivariate operator, we get the following Chebyshev-Grüss inequality.

Theorem 3.3.14. *If $f,g\in C(I)$, where $I=[0,1]\times[0,1]$ and $x,y\in[0,1]$ fixed, then the inequality*

$$\begin{aligned} &|T(f,g;x,y)| \\ &\le \frac{1}{4}\cdot\widetilde{\omega}_{d_2}\left(f;4\cdot\sqrt{S_{\Delta_{n_1},\Delta_{n_2}}(d_2^2(\cdot,(x,y));(x,y))}\right)\cdot\widetilde{\omega}_{d_2}\left(g;4\cdot\sqrt{S_{\Delta_{n_1},\Delta_{n_2}}(d_2^2(\cdot,(x,y));(x,y))}\right) \\ &\le \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;2\cdot\sqrt{\frac{1}{n_1^2}+\frac{1}{n_2^2}}\right)\cdot\widetilde{\omega}_{d_2}\left(g;2\cdot\sqrt{\frac{1}{n_1^2}+\frac{1}{n_2^2}}\right) \end{aligned}$$

holds.

3.3.2.9 Bivariate convolution operators

The second moment of the tensor product convolution operator is given by

$$G_{m(n_1),m(n_2)}(d_2^2(\cdot,(x_1,x_2));(x_1,x_2)) = G_{m(n_1)}((\cdot - x_1)^2;x_1) + G_{m(n_2)}((\cdot - x_2)^2;x_2)$$
$$= x_1^2\left\{\frac{3}{2} - 2\cdot\rho_{1,m(n_1)} + \frac{1}{2}\rho_{2,m(n_1)}\right\} + (1-x_1^2)\cdot\left\{\frac{1}{2} - \frac{1}{2}\rho_{2,m(n_1)}\right\}$$
$$+ x_2^2\left\{\frac{3}{2} - 2\cdot\rho_{1,m(n_2)} + \frac{1}{2}\rho_{2,m(n_2)}\right\} + (1-x_2^2)\cdot\left\{\frac{1}{2} - \frac{1}{2}\rho_{2,m(n_2)}\right\}$$

When considering different degrees $m(n_i)$, $i = 1,2$, we obtain different convolution type operators.

For example, if we let the degrees of the two operators be $m(n_1) = n_1 - 1$ and $m(n_2) = n_2 - 1$, for $n_1, n_2 \in \mathbb{N}$, we get the Fejér-Korovkin kernels, given by

$$K_{n_1-1}(v_1) = \frac{1}{n_1+1}\left(\frac{\sin\left(\frac{\pi}{n_1+1}\right)\cdot\cos\left((n_1+1)\cdot\frac{v_1}{2}\right)}{\cos(v_1) - \cos\left(\frac{\pi}{n_1+1}\right)}\right)^2,$$
$$K_{n_2-1}(v_2) = \frac{1}{n_2+1}\left(\frac{\sin\left(\frac{\pi}{n_2+1}\right)\cdot\cos\left((n_2+1)\cdot\frac{v_2}{2}\right)}{\cos(v_2) - \cos\left(\frac{\pi}{n_2+1}\right)}\right)^2,$$

with

$$\rho_{1,n_i-1} = \cos\left(\frac{\pi}{n_i+1}\right),$$
$$\rho_{2,n_i-1} = \frac{n_i}{n_i+1}\cos\left(\frac{2\pi}{n_i+1}\right) + \frac{1}{n_i+1},\ i = 1,2.$$

In this case the second moment can be estimated by

$$G_{n_1-1,n_2-1}(d_2^2(\cdot - (x_1,x_2));(x_1,x_2)) = G_{n_1-1}(d_2^2(\cdot - x_1);x_1) + G_{n_2-1}(d_2^2(\cdot - x_2);x_2)$$
$$\leq \left|\frac{3}{2} - 2\rho_{1,n_1-1} + \frac{1}{2}\rho_{2,n_1-1}\right| + \frac{1}{2}\left|1 - \rho_{2,n_1-1}\right|$$
$$+ \left|\frac{3}{2} - 2\rho_{1,n_2-1} + \frac{1}{2}\rho_{2,n_2-1}\right| + \frac{1}{2}\left|1 - \rho_{2,n_2-1}\right|$$
$$\leq \left(3\cdot\left(\frac{\pi}{n_1+1}\right)^2 + \left(\frac{\pi}{n_1+1}\right)^2\right) + \left(3\cdot\left(\frac{\pi}{n_2+1}\right)^2 + \left(\frac{\pi}{n_2+1}\right)^2\right)$$
$$\leq 4\left(\frac{\pi}{n_1+1}\right)^2 + 4\left(\frac{\pi}{n_2+1}\right)^2$$
$$= 4\pi^2\left(\frac{1}{(n_1+1)^2} + \frac{1}{(n_2+1)^2}\right).$$

Theorem 3.3.15. *For $f, g \in C(I)$, where $I = [-1,1]^2$, the Chebyshev-Grüss inequality for* *the tensor product convolution operator with the Fejér-Korovkin kernel is given by*

$$
\begin{aligned}
&|T(f,g;(x_1,x_2))| \\
&\leq \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;4\sqrt{4\pi^2\left(\frac{1}{(n_1+1)^2}+\frac{1}{(n_2+1)^2}\right)}\right)\cdot\widetilde{\omega}_{d_2}\left(g;4\sqrt{4\pi^2\left(\frac{1}{(n_1+1)^2}+\frac{1}{(n_2+1)^2}\right)}\right) \\
&= \frac{1}{4}\widetilde{\omega}_{d_2}\left(f;8\pi\sqrt{\frac{1}{(n_1+1)^2}+\frac{1}{(n_2+1)^2}}\right)\cdot\widetilde{\omega}_{d_2}\left(g;8\pi\sqrt{\frac{1}{(n_1+1)^2}+\frac{1}{(n_2+1)^2}}\right).
\end{aligned}
$$

Remark 3.3.16. For the other kernels, and other bivariate convolution-type operators, respectively, similar results can be obtained.

3.3.3 Bivariate Chebyshev-Grüss inequalities via discrete oscillations

3.3.3.1 Bivariate discrete (positive) linear functional case

In [58] the authors obtained a Chebyshev-Grüss-type inequality that involves oscillations of functions. We give here a generalization of the results obtained in [58], considering the bivariate discrete linear functional case. Such results were published in [4].

Let X be an arbitrary set and $B(I)$ the set of all real-valued, bounded functions on $I = X^2$. Take $a_n, b_n \in \mathbb{R}$, $n \geq 0$, such that $\sum_{n=0}^{\infty}|a_n| < \infty$, $\sum_{n=0}^{\infty} a_n = 1$ and $\sum_{n=0}^{\infty}|b_n| < \infty$, $\sum_{n=0}^{\infty} b_n = 1$, respectively. Furthermore, let $x_n \in X$, $n \geq 0$ and $y_m \in X$, $m \geq 0$ be arbitrary mutually distinct points. For $f \in B(I)$ set $f_{n,m} := f(x_n, y_m)$. Now consider the functional $L : B(I) \to \mathbb{R}$, $Lf = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m f_{n,m}$. Then L is linear and reproduces constant functions.

Theorem 3.3.17. *The Chebyshev-Grüss inequality for the above linear functional L is given by:*

$$
|L(fg) - L(f)\cdot L(g)| \leq \frac{1}{2} osc_L(f)\cdot osc_L(g)\cdot \sum_{n,m,i,j=0,(n,m)\neq(i,j)} |a_n b_m a_i b_j|,
$$

where $f, g \in B(I)$ and we define the oscillations to be:

$$
\begin{aligned}
osc_L(f) &:= \sup\{|f_{n,m} - f_{i,j}| : n,m,i,j \geq 0\}, \\
osc_L(g) &:= \sup\{|g_{n,m} - g_{i,j}| : n,m,i,j \geq 0\}.
\end{aligned}
$$

Proof. We have

$$L(fg) - L(f) \cdot L(g) = \sum_{n,m=0}^{\infty} a_n b_m f_{n,m} g_{n,m} - \sum_{n,m=0}^{\infty} a_n b_m f_{n,m} \cdot \sum_{i,j=0}^{\infty} a_i b_j g_{i,j}$$

$$= \sum_{n,m=0}^{\infty} \left(\sum_{i,j=0}^{\infty} a_i b_j \right) a_n b_m f_{n,m} g_{n,m} - \sum_{n,m=0}^{\infty} a_n b_m f_{n,m} \cdot \sum_{i,j=0}^{\infty} a_i b_j g_{i,j}$$

$$= \sum_{n,m=0}^{\infty} a_n^2 b_m^2 f_{n,m} g_{n,m} + \sum_{n,m=0}^{\infty} \left(\sum_{i,j=0,(i,j)\neq(n,m)}^{\infty} a_i b_j \right) a_n b_m f_{n,m} g_{n,m}$$

$$- \sum_{n,m=0}^{\infty} a_n^2 b_m^2 f_{n,m} g_{n,m} - \sum_{n,m=0}^{\infty} \left(\sum_{i,j=0,(i,j)\neq(n,m)}^{\infty} a_n b_m a_i b_j f_{n,m} g_{i,j} \right)$$

$$= \sum_{n,m,i,j=0,(i,j)\neq(n,m)}^{\infty} a_i b_j a_n b_m f_{n,m} (g_{n,m} - g_{i,j}).$$

The above identity can be written in the following way

$$L(fg) - L(f) \cdot L(g) = \sum_{n,m,i,j=0,(i,j)\neq(n,m)}^{\infty} a_n b_m a_i b_j f_{i,j} (g_{i,j} - g_{n,m}).$$

Therefore

$$2(L(fg) - L(f)L(g)) = \sum_{n,m,i,j=0,(i,j)\neq(n,m)}^{\infty} a_i b_j a_n b_m (f_{n,m} - f_{i,j})(g_{n,m} - g_{i,j}),$$

and the theorem is proven. □

Theorem 3.3.18. *In particular, if $a_n \geq 0$, $b_m \geq 0$, $n, m \geq 0$, then L is a positive linear functional and we have:*

$$|L(fg) - Lf \cdot Lg| \leq \frac{1}{2} \cdot \left(1 - \sum_{n=0}^{\infty} a_n^2 \cdot \sum_{m=0}^{\infty} b_m^2 \right) \cdot osc_L(f) \cdot osc_L(g),$$

for $f, g \in B(I)$ and the oscillations given as above.

Proof. In this case we have

$$\sum_{n,m,i,j=0,(n,m)\neq(i,j)}^{\infty} \left| a_n b_m a_i b_j \right| = \sum_{n,m=0}^{\infty} a_n b_m \sum_{i,j=0,(i,j)\neq(n,m)}^{\infty} a_i b_j$$

$$= \sum_{n,m=0}^{\infty} a_n b_m \left(\sum_{i,j=0}^{\infty} a_i b_j - a_n b_m \right) = \sum_{n,m=0}^{\infty} a_n b_m (1 - a_n b_m)$$

$$= \sum_{n,m=0}^{\infty} a_n b_m - \sum_{n,m=0}^{\infty} a_n^2 b_m^2 = 1 - \left(\sum_{n=0}^{\infty} a_n^2 \right) \cdot \left(\sum_{m=0}^{\infty} b_m^2 \right),$$

so the result follows as a consequence of Theorem 3.3.17. □

3.3.4 Applications to bivariate (positive) linear operators

3.3.4.1 Application for the bivariate Bernstein operator

According to Theorem 3.3.18, for each $x, y \in X$, $f, g \in B(I)$ we have

$$\begin{aligned} &|T(f,g;(x,y))| \\ &\leq \frac{1}{2}\left(1-\sum_{k_1=0}^{n_1} b_{n_1,k_1}^2(x)\cdot\sum_{k_2=0}^{n_2} b_{n_2,k_2}^2(y)\right)\cdot osc_{B_{n_1,n_2}}(f)\cdot osc_{B_{n_1,n_2}}(g), \end{aligned} \tag{3.3.3}$$

where

$$osc_{B_{n_1,n_2}}(f) := \max\{|f_{k,l}-f_{s,t}| : k,s=0,\dots,n_1; l,t=0,\dots,n_2\},$$

and $f_{k,l} := f\left(\frac{k}{n_1}, \frac{l}{n_2}\right)$; similar definitions apply to g.

Let $\varphi_{n_1}(x) := \sum\limits_{k_1=0}^{n_1} b_{n_1,k_1}^2(x)$, $x \in [0,1]$. Then we get

$$\varphi_{n_1}(x) \geq \frac{1}{n_1+1},\ x \in [0,1],$$

and the same holds for $\varphi_{n_2}(y)$, $y \in X$. Therefore it holds

$$\begin{aligned} &|T(f,g;(x,y))| \\ &\leq \frac{1}{2}\left(1-\frac{1}{n_1+1}\cdot\frac{1}{n_2+1}\right)\cdot osc_{B_{n_1,n_2}}(f)\cdot osc_{B_{n_1,n_2}}(g) \\ &= \frac{1}{2}\cdot\frac{n_2n_2+n_2+n_1}{(n_1+1)(n_2+1)}\cdot osc_{B_{n_1,n_2}}(f)\cdot osc_{B_{n_1,n_2}}(g), \end{aligned} \tag{3.3.4}$$

for $x, y \in X$.

Remark 3.3.19. We have seen in the univariate case (see (2.2.15) in a previous section), and it was also proved in [58] that

$$\varphi_{n_1}(x) \geq \frac{1}{4^{n_1}}\binom{2n_1}{n_1},\ x \in X,$$

with equality if and only if $x = \frac{1}{2}$. A similar result holds for $\varphi_{n_2}(y)$, for $y \in X$.

Theorem 3.3.20. *The Chebyshev-Grüss inequality for the bivariate Bernstein operator is:*

$$\begin{aligned} &|T(f,g;(x,y))| \\ &\leq \frac{1}{2}\left(1-\binom{2n_1}{n_1}\binom{2n_2}{n_2}\frac{1}{4^{n_1}}\cdot\frac{1}{4^{n_2}}\right)\cdot osc_{B_{n_1,n_2}}(f)\cdot osc_{B_{n_1,n_2}}(g),\ x,y \in [0,1]. \end{aligned} \tag{3.3.5}$$

In comparison to the Chebyshev-Grüss inequality for the bivariate Bernstein operator, given in Theorem 3.3.3, we make some observations in the next remark.

Remark 3.3.21. In (3.3.3) and (3.3.1), the right-hand side depends on (x, y) and vanishes when $(x, y) \to (i, j)$ for $i, j \in \{0, 1\}$. The maximum value of it, as a function of (x, y), is attained for $x = y = \frac{1}{2}$, and (3.3.4), (3.3.5), (3.3.2) illustrate this fact. On the other hand, in (3.3.3) the oscillations of f and g are relative only to the points $\left(\frac{k}{n_1}, \frac{l}{n_2}\right)$, $0 \leq k \leq n_1$, $0 \leq l \neq n_2$ while in (3.3.1) the oscillations, expressed in terms of $\widetilde{\omega}$, are relative to the whole interval I.

3.3.4.2 Application for $S_{\Delta_{n_1},\Delta_{n_2}}$

Using Theorem 3.3.18, we get an inequality of the form

$$\begin{aligned}&\left|S_{\Delta_{n_1},\Delta_{n_2}}(f\cdot g)(x,y)-S_{\Delta_{n_1},\Delta_{n_2}}f(x,y)\cdot S_{\Delta_{n_1},\Delta_{n_2}}g(x,y)\right|\\&\leq\frac{1}{2}\left(1-\sum_{k_1=0}^{n_1}u_{n_1,k_1}^2(x)\cdot\sum_{k_2=0}^{n_2}u_{n_2,k_2}^2(y)\right)\cdot osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(f)\cdot osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(g),\end{aligned}$$

for each $x,y\in[0,1]$, $f,g\in B(I)$, where

$$osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(f):=\max\{|f_{s,l}-f_{r,t}|:0\leq s,r\leq n_1,0\leq l,t\leq n_2\}$$

and $f_{s,l}:=f(\frac{s}{n_1},\frac{l}{n_2})$; similar definition applies to g and its oscillation.

In this case, we need to find the minimum of the sums $\tau_{n_1}(x):=\sum_{k_1=0}^{n_1}u_{n_1,k_1}^2(x)$ and $\tau_{n_2}(y):=\sum_{k_2=0}^{n_2}u_{n_2,k_2}^2(y)$. For particular intervals $x\in\left[\frac{k_1-1}{n_1},\frac{k_1}{n_1}\right]$ and $y\in\left[\frac{k_2-1}{n_2},\frac{k_2}{n_2}\right]$, we get that

$$\begin{aligned}\tau_{n_1}(x)&:=\sum_{k_1=0}^{n_1}u_{n_1,k_1}^2(x)\\&=(n_1x-k_1+1)^2+(k_1-n_1x)^2,\ \text{for } k_1=1,\ldots,n_1\end{aligned}$$

and something similar for $\tau_{n_2}(y)$. The functions $\tau_{n_1}(x)$ and $\tau_{n_2}(y)$ are minimal if and only if $x=\frac{2k_1-1}{2n_1}$, $y=\frac{2k_2-1}{2n_2}$ and the minimum value for both $\tau_{n_1}(x)$ and $\tau_{n_2}(y)$ is $\frac{1}{2}$.

Theorem 3.3.22. *The Chebyshev-Grüss inequality for $S_{\Delta_{n_1},\Delta_{n_2}}$ is*

$$\begin{aligned}&|T(f,g;(x,y))|\\&\leq\frac{1}{2}\left(1-\sum_{k_1=0}^{n_1}u_{n_1,k_1}^2(x)\cdot\sum_{k_2=0}^{n_2}u_{n_2,k_2}^2(y)\right)\cdot osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(f)\cdot osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(g)\\&\leq\frac{1}{2}\left(1-\frac{1}{4}\right)\cdot osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(f)\cdot osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(g)\\&\leq\frac{3}{8}\cdot osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(f)\cdot osc_{S_{\Delta_{n_1},\Delta_{n_2}}}(g).\end{aligned}$$

3.3.4.3 Application for bivariate special King operators V_{n_1,n_2}^*

We want to find the infimum of the sums

$$\varphi_{n_1}(x):=\sum_{k_1=0}^{n_1}(v_{n_1,k_1}^*(x))^2$$

and

$$\varphi_{n_2}(y):=\sum_{k_2=0}^{n_2}(v_{n_2,k_2}^*(y))^2.$$

We consider the following special cases.

For $n_1 = n_2 = 1$, we have

$$\begin{aligned}\varphi_1(x) &= \sum_{k_1=0}^{1} (v^*_{1,k_1}(x))^2 \\ &= (v^*_{1,0}(x))^2 + (v^*_{1,1}(x))^2 \\ &= 2x^4 - 2x^2 + 1,\end{aligned}$$

and the same thing for $\varphi_1(y)$. These are minimum for $x, y = \frac{\sqrt{2}}{2}$ and the minimum values are in both cases $\varphi_1\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}$.

Theorem 3.3.23. *The Chebyshev-Grüss inequality via discrete oscillations in the bivariate case, for $n_1, n_2 = 1$, is given by*

$$\begin{aligned}|T(f,g;(x,y))| &\le \frac{3}{8} \cdot osc_{V^*_{1,1}}(f) \cdot osc_{V^*_{1,1}}(g) \\ &= \frac{3}{8} \cdot \max\{|f_{k,l} - f_{s,t}| : 0 \le k, s \le 1; 0 \le l, t \le 1\}.\end{aligned}$$

In the case $n_1, n_2 = 2, 3, \ldots, n_1 \ne n_2$, we have

$$\begin{aligned}\varphi_{n_1}(x) &= \sum_{k_1=0}^{n_1} (v^*_{n_1,k_1}(x))^2 \\ &= \sum_{k_1=0}^{n_1} \binom{n_1}{k_1}^2 (r^*_{n_1}(x))^2 k_1 (1 - r^*_{n_1}(x) 2(n_1 - k_1)),\end{aligned}$$

and a similar result for $\varphi_{n_2}(y)$, which are both more difficult to estimate. What we can say for sure is that

$$\varphi_{n_1}(x) \ge \frac{1}{n_1 + 1}$$

and

$$\varphi_{n_2}(y) \ge \frac{1}{n_2 + 1}$$

hold, for $x, y \in [0,1]$ and $n_1, n_2 = 2, 3, \ldots$. So we obtain

$$1 - \sum_{k_1=0}^{n_1} (v^*_{n_1,k_1}(x))^2 \le \frac{n_1}{n_1 + 1},$$

and an analogous inequality for $1 - \sum_{k_2=0}^{n_2} (v^*_{n_2,k_2}(y))^2$, so we get the following result.

Theorem 3.3.24. *The Chebyshev-Grüss inequality for the bivariate King operators via discrete oscillations, for $n_1, n_2 = 2, 3, \ldots$, is given by*

$$\begin{aligned}|T(f,g;(x,y))| &\le \frac{1}{2}\left(1 - \frac{1}{n_1+1} \cdot \frac{1}{n_2+1}\right) \cdot osc_{V^*_{n_1,n_2}}(f) \cdot osc_{V^*_{n_1,n_2}}(g) \\ &= \frac{1}{2} \frac{n_2 n_1 + n_2 + n_1}{(n_1+1)(n_2+1)} \cdot osc_{V^*_{n_1,n_2}}(f) \cdot osc_{V^*_{n_1,n_2}}(g),\end{aligned}$$

where

$$osc_{V^*_{n_1,n_2}}(f) := \max\left\{|f_{s,l} - f_{r,t}| : 0 \le s, r \le n_1, 0 \le l, t \le n_2, \right\},$$

where $f_{s,l} := f\left(\frac{s}{n_1}, \frac{l}{n_2}\right)$ and a similar definition can be applied to g.

3.3.4.4 Application for the bivariate Mirakjan-Favard-Szász operators

We want to find the infimum of the sums

$$\sigma_{n_1}(x) := e^{-2n_1 x} \sum_{k_1=0}^{\infty} \frac{(n_1 x)^{2k_1}}{(k_1!)^2}$$

and

$$\sigma_{n_2}(y) := e^{-2n_2 y} \sum_{k_2=0}^{\infty} \frac{(n_2 y)^{2k_2}}{(k_2!)^2}$$

Theorem 3.3.25. *The Chebyshev-Grüss inequality via discrete oscillations in the bivariate case is given by*

$$|T(f,g;(x,y))| \le \frac{1}{2} \cdot (1 - \sigma_{n_1}(x) \cdot \sigma_{n_2}(y)) \cdot osc_{M_{n_1,n_2}}(f) \cdot osc_{M_{n_1,n_2}(g)},$$

where $f, g \in C_b([0,\infty) \times [0,\infty))$, $osc_{M_{n_1,n_2}}(f) := \sup\{|f_{s,l} - f_{r,t}| : 0 \le s, r < \infty, 0 \le l, t < \infty\}$, *with* $f_{s,l} := f\left(\frac{s}{n_1}, \frac{l}{n_2}\right)$. *A similar definition is applied to the second function.* $C_b([0,\infty) \times [0,\infty))$ *is the set of all continuous, real-valued, bounded functions on* $[0,\infty) \times [0,\infty)$.

We have seen in the univariate case that $\inf_{x \ge 0} \sigma_{n_1}(x) = 0$ and $\inf_{y \ge 0} \sigma_{n_2}(y) = 0$. Then the above inequality looks as follows:

Theorem 3.3.26. *The Chebyshev-Grüss inequality via discrete oscillations for the bivariate Mirakjan-Favard-Szász operator becomes*

$$|T(f,g;(x,y))| \le \frac{1}{2} osc_{M_{n_1,n_2}}(f) \cdot osc_{M_{n_1,n_2}}(g),$$

where the functions f and g and the oscillations are given as above.

3.3.4.5 Application for the bivariate Baskakov operators

We set

$$\vartheta_{n_1}(x) := \frac{1}{(1+x)^{2n_1}} \sum_{k_1=0}^{\infty} \binom{n_1 + k_1 - 1}{k_1}^2 \left(\frac{x}{1+x}\right)^{2k_1}, \text{ for } x \ge 0,$$

and

$$\vartheta_{n_2}(y) := \frac{1}{(1+y)^{2n_2}} \sum_{k_2=0}^{\infty} \binom{n_2 + k_2 - 1}{k_2}^2 \left(\frac{y}{1+y}\right)^{2k_2}, \text{ for } y \ge 0.$$

and we need to find $\inf\limits_{x \ge 0} \vartheta_{n_1}(x)$ and $\inf\limits_{y \ge 0} \vartheta_{n_2}(y)$.

We have the following result:

Theorem 3.3.27. *The Chebyshev-Grüss inequality via discrete oscillations for the Baskakov operator in the bivariate case is given by*

$$|T(f,g;(x,y))| \le \frac{1}{2} \cdot (1 - \vartheta_{n_1}(x) \cdot \vartheta_{n_2}(y)) \cdot osc_{A_{n_1,n_2}}(f) \cdot osc_{A_{n_1,n_2}(g)},$$

where $f, g \in C_b([0,\infty) \times [0,\infty))$*,* $osc_{A_{n_1,n_2}}(f) := sup\{|f_{s,l} - f_{r,t}| : 0 \leq s, r < \infty, 0 \leq l, t < \infty\}$*, with* $f_{s,l} := f\left(\frac{s}{n_1}, \frac{l}{n_2}\right)$*. A similar definition is applied to the second function.* $C_b([0,\infty) \times [0,\infty))$ *is the set of all continuous, real-valued, bounded functions on* $[0,\infty) \times [0,\infty)$*.*

From the univariate case we know that $\inf_{x\geq 0} \vartheta_{n_1}(x) = 0$ and $\inf_{y\geq 0} \vartheta_{n_2}(y) = 0$. Then the above inequality looks as follows.

Theorem 3.3.28. *The following inequality*

$$|T(f,g;(x,y))| \leq \frac{1}{2} osc_{A_{n_1,n_2}}(f) \cdot osc_{A_{n_1,n_2}}(g)$$

holds, where the functions f and g and the oscillations are given as above.

3.3.4.6 Application for the bivariate Lagrange operators

We only give an inequality using these special oscillations for the bivariate Lagrange operator. In the case of oscillations involving the least concave majorant of the modulus of continuity, the inequalities are more complicated.

Theorem 3.3.29. *The Chebyshev-Grüss inequality via discrete oscillations for the bivariate Lagrange operator is given by*

$$\begin{aligned} &|T(f,g;(x,y))| \\ &\leq \frac{1}{2}\left(\Lambda_{n_1}^2(x) \cdot \Lambda_{n_2}^2(y) - \frac{1}{16}\right) \cdot osc_{L_{n_1,n_2}}(f) \cdot osc_{L_{n_1,n_2}}(g), \end{aligned}$$

where $f, g \in B(I)$. The oscillation for f is defined by

$$osc_{L_{n_1,n_2}}(f) := max\{|f(x_{k_1,n_1}, y_{k_2,n_2}) - f(x_{m_1,n_1}, y_{m_2,n_2})| : 1 \leq k_1, m_1 \leq n_1, 1 \leq k_2, m_2 \leq n_2\}.$$

A similar definition is given for the oscillation of the second function.

Proof. We have

$$\begin{aligned} &\sum_{k_1,m_1=1}^{n_1}\left(\sum_{k_2,m_2=1,(k_1,k_2)\neq(m_1,m_2)}^{n_2} |l_{k_1,n_1}(x)l_{k_2,n_2}(y)l_{m_1,n_1}(x)l_{m_2,n_2}(y)|\right) \\ &= \left(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\left|l_{i,n_1}(x)l_{j,n_2}(y)\right|\right)^2 - \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} l_{i,n_1}^2(x)l_{j,n_2}^2(y) \\ &= \Lambda_{n_1}^2(x)\Lambda_{n_2}^2(y) - \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} l_{i,n_1}^2(x)l_{j,n_2}^2(y) \\ &\leq \Lambda_{n_1}^2(x)\Lambda_{n_2}^2(y) - \frac{1}{16}, \end{aligned}$$

and the theorem is proven. □

For the Lagrange operator based upon Chebyshev nodes, the inequality looks a bit more complicated. We have

Theorem 3.3.30. *The relationship*

$$
\begin{aligned}
&|T(f,g;(x,y))| \leq \frac{1}{2} \cdot osc_{L_{n_1,n_2}}(f) \cdot osc_{L_{n_1,n_2}}(g) \\
&\cdot \left[\Lambda_{n_1}^2(x)\Lambda_{n_2}^2(y) - c\left(1 + (\cos n_1 t_1)^2 \cdot \frac{\pi^2}{6}\right) \cdot \left(1 + (\cos n_2 t_2)^2 \cdot \frac{\pi^2}{6}\right)\right]
\end{aligned}
$$

holds, for $f, g \in B(I)$, $x = \cos t_1$, $y = \cos t_2$ *and suitable constant* c.

4 Univariate Ostrowski Inequalities

4.1 Auxiliary and historical results

One of Ostrowski's classical inequalities deals with the most primitive form of a quadrature rule. It was published in 1938 in Switzerland (see [89]) and reads in its original form as follows.

Theorem 4.1.1. *Es sei $h(x)$ im Intervall $J : a < x < b$ stetig und differentiierbar, und es sei in J durchweg*

$$\left|h'(x)\right| \leq m,\ m > 0.$$

Dann gilt für jedes x aus J:

$$\left|h(x) - \frac{1}{b-a}\int_a^b h(x)\mathrm{d}x\right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)m.$$

A simplified form can be found, for example, in G. Anastassiou's 1995 article [8].

Theorem 4.1.2. *Let f be in $C^1[a,b]$, $x \in [a,b]$. Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \leq \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \cdot \left\|f'\right\|_\infty.$$

The characteristic feature of Ostrowski's approach is thus to approximate an integral by a single value of the function in question and to estimate the difference assuming differentiability of the function in question. The latter is a dispensable assumption, as was observed by A. Acu and H. Gonska in the next result (see [1]).

Theorem 4.1.3. *Let $L : C[a,b] \to C[a,b]$ be non-zero, linear and bounded, and such that $L : C^1[a,b] \to C^1[a,b]$ with $\|(Lg)'\| \leq c_L \cdot \|g'\|$ for all $g \in C^1[a,b]$. Then for all $f \in C[a,b]$ and $x \in [a,b]$ we have*

$$\left|Lf(x) - \frac{1}{b-a}\int_a^b Lf(t)\mathrm{d}t\right| \leq \|L\| \cdot \tilde{\omega}\left(f; \frac{c_L}{\|L\|} \cdot \frac{(x-a)^2 + (b-x)^2}{2(b-a)}\right).$$

The right hand side in the latter inequality is given in terms of the least concave majorant of the first order modulus of continuity of an arbitrary $f \in C[a,b]$ and thus generalizes Ostrowski's inequality in the form given by Anastassiou.

We remark that - according to our knowledge - B. and I. Gavrea in [40] were the first to observe the possibility of using "omega-tilde" in this context.

Ostrowski inequalities have attracted a most remarkable amount of attention in the past. The reader should consult Ch. XV on "Integral inequalities involving functions with bounded derivatives" in the book by D.S. Mitrinović et al. [84] and Chapters 2 - 9 in the recent monography of G. Anastassiou [9].

In this chapter we present a generalization of the above-mentioned result of Acu et al. for integrals w.r.t. probability measures λ and apply the new estimates to iterates of certain positive linear operators and to differences of such mappings (see [55]).

We first consider the space

$$Lip[0,1] := \left\{ f \in C[0,1] \,|\, |f|_{Lip} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|} < \infty \right\}.$$

Let $M_1^+[0,1]$ be the set of all probability Borel measures on $[0,1]$ and $\lambda \in M_1^+[0,1]$ a given measure. Then the inequality

$$\begin{aligned} \left| f(x) - \int_0^1 f(t) \mathrm{d}\lambda(t) \right| &= \left| \int_0^1 (f(x) - f(t)) \mathrm{d}\lambda(t) \right| \\ &\leq \int_0^1 |f(x) - f(t)| \, \mathrm{d}\lambda(t) \\ &\leq |f|_{Lip} \int_0^1 |t - x| \, \mathrm{d}\lambda(t) \end{aligned}$$

holds, for $f \in Lip[0,1]$ and $x \in [0,1]$. Let also

$$w_\lambda(x) := \int_0^1 |t-x| \, \mathrm{d}\lambda(t).$$

We have obtained a general form of Ostrowski's inequality:

$$\left| f(x) - \int_0^1 f(t) \mathrm{d}\lambda(t) \right| \leq |f|_{Lip} \, w_\lambda(x), \tag{4.1.1}$$

for all $f \in \mathrm{Lip}[0,1]$, $\lambda \in M_1^+[0,1]$ and $x \in [0,1]$.

4.2 Over-iterates of positive linear operators

If we consider any positive linear operator $L_n : C[0,1] \to C[0,1]$, then the powers of L_n are defined inductively by

$$L_n^0 := Id, \; L_n^1 := L_n \; \text{ and } L_n^{m+1} := L_n \circ L_n^m, \; m \in \mathbb{N}.$$

When talking about iterates, the interest is in studying the behaviour of the powers of the operator L_n, when taking the case n fixed and m going to infinity. This is why the operators are called *over-iterated*.

More methods can be described when we want to consider the over-iteration of our operators L_n. Among such methods, P. Piţul studied three of them in her PhD Thesis [93], namely: the contraction principle, a general quantitative method and one method involving spectral properties of the operator. For details and different results with respect to such over-iterates, see [93] and [96].

4.3 A result of A. Acu and H. Gonska

We shall present a slight modification of the inequality established by A. Acu and H. Gonska in [1].

Theorem 4.3.1. *Let $L : C[0,1] \to C[0,1]$ be non-zero, linear and bounded. Suppose that $L(Lip[0,1]) \subset Lip[0,1]$ and there exists $c_L > 0$ such that*

$$|Lg|_{Lip} \leq c_L \, |g|_{Lip} \, ,$$

for all $g \in Lip[0,1]$. Then for all $f \in C[0,1]$, $\lambda \in M_1^+[0,1]$ and $x \in [0,1]$ we have

$$\left| Lf(x) - \int_0^1 Lf(t) \mathrm{d}\lambda(t) \right| \leq \|L\| \, \widetilde{\omega} \left(f; \frac{c_L}{\|L\|} w_\lambda(x) \right).$$

Proof. Let $A_x : C[0,1] \to \mathbb{R}$ be defined by

$$A_x(f) := f(x) - \int_0^1 f(t) \mathrm{d}\lambda(t).$$

Then A_x is a bounded linear functional with $\|A_x\| \leq 2$. We have

$$\begin{aligned} |A_x(Lf)| &\leq |Lf(x)| + \int_0^1 |Lf(t)| \, \mathrm{d}\lambda(t) \\ &\leq 2 \, \|L\| \, \|f\|_\infty \, , \end{aligned}$$

for $f \in C[0,1]$. Let $g \in Lip[0,1]$. By using (4.1.1) we get

$$|A_x(Lg)| = \left| Lg(x) - \int_0^1 Lg(t) \mathrm{d}\lambda(t) \right| \leq |Lg|_{Lip} \, w_\lambda(x) \leq c_L \, |g|_{Lip} \, w_\lambda(x).$$

Consequently,

$$\begin{aligned} |A_x(Lf)| &= |(A_x \circ L)(f - g + g)| \\ &\leq |(A_x \circ L)(f - g)| + |A_x(Lg)| \\ &\leq 2 \, \|L\| \, \|f - g\|_\infty + c_L \, |g|_{Lip} \, w_\lambda(x). \end{aligned}$$

Passing to the infimum over $g \in Lip[0,1]$ we get

$$\begin{aligned} |A_x(Lf)| &\leq 2 \, \|L\| \inf_{g \in Lip[0,1]} \left\{ \|f - g\|_\infty + \frac{c_L}{2 \, \|L\|} w_\lambda(x) \, |g|_{Lip} \right\} \\ &= \|L\| \, \widetilde{\omega} \left(f; \frac{c_L}{\|L\|} w_\lambda(x) \right). \end{aligned}$$

□

Corollary 4.3.2. *In the setting of Theorem 4.3.1 suppose that, moreover, L is a positive linear operator reproducing the constant functions. Then*

$$\left| Lf(x) - \int_0^1 Lf(t) \mathrm{d}\lambda(t) \right| \leq \widetilde{\omega} \left(f; c_L w_\lambda(x) \right)$$

holds, for all $f \in C[0,1]$, $\lambda \in M_1^+[0,1]$ and $x \in [0,1]$.

Let $e_0(x) := 1$ for $x \in [0,1]$. It is well known (see, e.g., [72], p.178) that if L is a positive linear operator and $Le_0 = e_0$, then L has at least one *invariant measure* μ, i.e., there exists $\mu \in M_1^+[0,1]$ such that

$$\int_0^1 Lf(t)\mathrm{d}\mu(t) = \int_0^1 f(t)\mathrm{d}\mu(t),$$

for $f \in C[0,1]$.

Now from Corollary 4.3.2 we obtain

Corollary 4.3.3. *Let $L : C[0,1] \to C[0,1]$ be a positive linear operator with $Le_0 = e_0$, and μ an invariant measure for L. Suppose that $L(Lip[0,1]) \subset Lip[0,1]$ and there exists $c_L > 0$ such that $|Lg|_{Lip} \le c_L\, |g|_{Lip}$, $g \in Lip[0,1]$. Then the inequality*

$$\left| Lf(x) - \int_0^1 f(t)\mathrm{d}\mu(t) \right| \le \widetilde{\omega}\left(f; c_L w_\mu(x)\right)$$

holds, for all $f \in C[0,1]$, $x \in [0,1]$.

Under the hypothesis of Corollary 4.3.3, let $m \ge 1$ be an integer. Then $L^m e_0 = e_0$, $|L^m g|_{Lip} \le c_L^m\, |g|_{Lip}$, $g \in Lip[0,1]$, and μ is an invariant measure for the iterate L^m. Consequently, we can state the following result.

Corollary 4.3.4. *In the setting of Corollary 4.3.3 we have*

$$\left| L^m f(x) - \int_0^1 f(t)\mathrm{d}\mu(t) \right| \le \widetilde{\omega}\left(f; c_L^m w_\mu(x)\right), \tag{4.3.1}$$

for all $f \in C[0,1]$, $x \in [0,1]$, $m \ge 1$. Moreover, if $c_L < 1$, then

$$\lim_{m\to\infty} L^m f = \left(\int_0^1 f(t)\mathrm{d}\mu(t) \right) e_0, \text{ uniformly on } [0,1],$$

and, consequently, L has exactly one invariant measure $\mu \in M_1^+[0,1]$.

Related results, in a more general context, can be found in [7].

4.4 Applications involving iterates of positive linear operators

Application 4.4.1. Let $n \ge 1$, $p = [n/2]$, $0 \le k \le p$, $0 \le x \le 1$. Consider the polynomials

$$\begin{aligned} w_{n,k}(x) :=& \frac{n+1-2p+2k}{(n+1)2^{n+1}x}\binom{n+1}{p-k} \\ &\cdot \left((1-x)^{p-k}(1+x)^{n+1-p+k} - (1-x)^{n+1-p+k}(1+x)^{p-k} \right). \end{aligned}$$

The operators $\beta_n : C[0,1] \to C[0,1]$, defined by

$$\beta_n f(x) := \sum_{k=0}^{p} f\left(\frac{n-2p+2k}{n}\right) w_{n,k}(x)$$

were introduced in [105] (see also [94], [95]). They are positive linear operators with $\beta_n e_0 = e_0$. According to the results of [94], $c_{\beta_n} = \frac{n-1}{n}$ and the probability measure concentrated on 1 is invariant for β_n. Now Corollary 4.3.4 entails

$$|\beta_n^m f(x) - f(1)| \le \widetilde{\omega}\left(f; \left(\frac{n-1}{n}\right)^m (1-x)\right),$$

for all $m, n > 1$, $f \in C[0,1]$, $x \in [0,1]$. This result supplements the qualitative results presented in ([95], Ex. 5.7.).

Application 4.4.2. For $n \ge 1$ and $j \in \{0, 1, \dots, n\}$ let

$$b_{n,j}(x) := \binom{n}{j} x^j (1-x)^{n-j}, \ x \in [0,1].$$

Let $0 \le \beta \le \gamma$, $\gamma > 0$. Consider the Stancu operators $S_n^{<0,\beta,\gamma>} : C[0,1] \to C[0,1]$ given by

$$S_n^{<0,\beta,\gamma>}(f;x) := \sum_{j=0}^{n} b_{n,j}(x) f\left(\frac{j+\beta}{n+\gamma}\right).$$

It is easy to verify that $c_{S_n^{<0,\beta,\gamma>}} = \frac{n}{n+\gamma} < 1$.

According to Corollary 4.3.4, L_n has a unique invariant measure $\mu_n \in M_1^+[0,1]$; in fact, μ_n was already determined in [53] and [99]. The quantitative result derived from (4.3.1) accompanies the qualitative results of [53] and [99]. In particular, we see that the rate of convergence, generally expressed by $c^m_{S_n^{<0,\beta,\gamma>}}$, is expressed here by $\left(\frac{n}{n+\gamma}\right)^m$.

Application 4.4.3. Consider the Bernstein-Durrmeyer operators with Jacobi weights $\alpha, \beta > -1$, defined by

$$M_n^{\alpha,\beta} f(x) := \sum_{j=0}^{n} b_{n,j}(x) \left(\int_0^1 t^{j+\alpha}(1-t)^{n-j+\beta} f(t) \mathrm{d}t\right) \Big/ \left(\int_0^1 t^{j+\alpha}(1-t)^{n-j+\beta} \mathrm{d}t\right),$$

for $f \in C[0,1]$, $x \in [0,1]$, $n \ge 1$.

According to the results of [7],

$$c_{M_n^{\alpha,\beta}} = \frac{n}{n+\alpha+\beta+2} < 1,$$

and the invariant measure μ is described by

$$\int_0^1 f(t) \mathrm{d}\mu(t) = \left(\int_0^1 t^\alpha (1-t)^\beta f(t) \mathrm{d}t\right) \Big/ \left(\int_0^1 t^\alpha (1-t)^\beta \mathrm{d}t\right), \text{ with } f \in C[0,1].$$

Using the Cauchy-Schwarz inequality we get

$$\begin{aligned} w_\mu(x) &= \int_0^1 |t-x| \, \mathrm{d}\mu(t) \le \left(\int_0^1 (t-x)^2 \mathrm{d}\mu(t)\right)^{\frac{1}{2}} \\ &= \left(\left(x - \frac{\alpha+1}{\alpha+\beta+2}\right)^2 + \frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^2(\alpha+\beta+3)}\right)^{\frac{1}{2}}. \end{aligned}$$

Now Corollary 4.3.4 entails

$$\left|(M_n^{\alpha,\beta})^m f(x) - \left(\int_0^1 t^\alpha (1-t)^\beta f(t)\mathrm{dt}\right) \Big/ \left(\int_0^1 t^\alpha (1-t)^\beta \mathrm{dt}\right)\right|$$
$$\le \tilde{\omega}\left(f; \left(\frac{n}{n+\alpha+\beta+2}\right)^m \left(\left(x - \frac{\alpha+1}{\alpha+\beta+2}\right)^2 + \frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^2(\alpha+\beta+3)}\right)^{\frac{1}{2}}\right)$$

This is a quantitative companion to the results of ([7], Section 3.2.).

Application 4.4.4. For each $n \ge 1$, let $\vartheta_n \in L^1[0,1]$, $\vartheta_n \ge 0$, be a periodic function with period $\frac{1}{n+1}$, such that

$$\int_0^{\frac{1}{n+1}} \vartheta_n(t)\mathrm{dt} = 1.$$

Consider the generalized Kantorovich operators $K_n : C[0,1] \to C[0,1]$ defined by

$$K_n f(x) := \sum_{j=0}^n b_{n,j}(x) \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(t)\vartheta_n(t)\mathrm{dt}.$$

It is easy to verify that $K_n(Lip[0,1]) \subset Lip[0,1]$ and $c_{K_n} = \frac{n}{n+1}$, $n \ge 1$.

We want to determine the invariant measure. Consider the matrix

$$T_n := \left[\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} b_{n,i}(t)\vartheta_n(t)\mathrm{dt}\right]_{i,j=0,1,\ldots,n}$$

and let $a = (a_0, a_1, \ldots, a_n)^t \in \mathbb{R}^{n+1}$. T_n is the transpose of a regular stochastic matrix, so that the system $T_n a = a$ has a unique solution with $a_i \ge 0$, $i = 0, 1, \ldots, n$, and $a_0 + \ldots + a_n = 1$. For this solution, we have

$$a_i = \sum_{j=0}^n a_j \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} b_{n,i}(t)\vartheta_n(t)\mathrm{dt},$$

for $i = 0, \ldots, n$. Let $\varphi_n(t) := a_j \vartheta_n(t)$, $t \in \left(\frac{j}{n+1}, \frac{j+1}{n+1}\right)$, $j = 0, \ldots, n$. Define $\mu_n \in M_1^+[0,1]$ by $d\mu_n(t) = \varphi_n(t)dt$. Then μ_n is the invariant measure of K_n. Indeed,

$$\begin{aligned}
\int_0^1 K_n f(t)\mathrm{d}\mu_n(t) &= \sum_{i=0}^n \int_0^1 b_{n,i}(t)\mathrm{d}\mu_n(t) \cdot \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} f(t)\vartheta_n(t)\mathrm{d}t \\
&= \sum_{i=0}^n \left(\sum_{j=0}^n \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} b_{n,i}(t) a_j \vartheta_n(t)\mathrm{dt}\right) \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} f(t)\vartheta_n(t)\mathrm{dt} \\
&= \sum_{i=0}^n a_i \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} f(t)\vartheta_n(t)\mathrm{dt} \\
&= \sum_{i=0}^n \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} f(t)\varphi_n(t)\mathrm{dt} \\
&= \int_0^1 f(t)\mathrm{d}\mu_n(t).
\end{aligned}$$

i) As a particular case, let $\alpha,\ \beta > -1$ and

$$\vartheta_n(t) = \left(t - \frac{j}{n+1}\right)^{\alpha} \left(\frac{j+1}{n+1} - t\right)^{\beta} \Big/ \left(\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} \left(s - \frac{j}{n+1}\right)^{\alpha} \left(\frac{j+1}{n+1} - s\right)^{\beta} \mathrm{d}s \right),$$

for all $t \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$, $j = 0, 1, \dots, n$. Denote the corresponding operators K_n by $K_n^{\alpha,\beta}$; it is easy to see that they can be expressed also as

$$K_n^{\alpha,\beta} f(x) = \frac{1}{B(\alpha+1, \beta+1)} \sum_{j=0}^{n} b_{n,j}(x) \int_0^1 s^{\alpha}(1-s)^{\beta} f\left(\frac{s+j}{n+1}\right) \mathrm{d}s.$$

Here $B(\cdot, \star)$ is the Beta function. In fact, these are the operators introduced in ([77], (1.5)).

ii) More particularly,

$$\begin{aligned} K_n^{0,0} f(x) &= (n+1) \sum_{j=0}^{n} b_{n,j}(x) \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(t) \mathrm{d}t \\ &= \sum_{j=0}^{n} b_{n,j}(x) \int_0^1 f\left(\frac{s+j}{n+1}\right) \mathrm{d}s \end{aligned}$$

are the classical Kantorovich operators. For them, as in the above general case, the parameter c_{K_n} is $\frac{n}{n+1}$; moreover, the invariant measure μ is the Lebesgue measure on $[0, 1]$. Thus

$$w_{\mu}(x) = \int_0^1 |t - x| \, \mathrm{d}t = \left(x - \frac{1}{2}\right)^2 + \frac{1}{4}.$$

From Corollary 4.3.4 we infer

$$\left| (K_n^{0,0})^m f(x) - \int_0^1 f(t) \mathrm{d}t \right| \le \tilde{\omega} \left(f; \left(\frac{n}{n+1}\right)^m \left(\left(x - \frac{1}{2}\right)^2 + \frac{1}{4} \right) \right).$$

Related results, for multivariate Kantorovich operators, can be found in ([7], Section 3.1).

Remark 4.4.5. Results of this type are especially significant in case of sequences (L_n) for which the strong limit

$$T(t) := \lim_{n \to \infty} L_n^{[nt]}$$

exists for all $t \ge 0$. In such a case $(T(t))_{t \ge 0}$ is a C_0-semigroup of operators and the above results can be used in order to study its asymptotic behaviour from a quantitative point of view. Details can be found in [7].

4.5 Applications involving differences of positive linear operators

In the preceding section the main tool was Corollary 4.3.4 which makes use of the invariant measure. Now we present applications of Corollary 4.3.2.

Let again $L : C[0,1] \to C[0,1]$ be a positive linear operator with $Le_0 = e_0$. Suppose that $L(Lip[0,1]) \subset Lip[0,1]$ and there exists $c_L > 0$ such that $|Lg|_{Lip} \le c_L |g|_{Lip}$, for all $g \in Lip[0,1]$. Let $A : C[0,1] \to C[0,1]$ be a positive linear operator with $Ae_0 = e_0$. For each $x \in [0,1]$ consider the measure $\lambda_x \in M_1^+[0,1]$ defined by

$$\int_0^1 f(t)\mathrm{d}\lambda_x(t) = Af(x),\ f \in C[0,1].$$

Then we have

$$\int_0^1 Lf(t)\mathrm{d}\lambda_x(t) = A(Lf)(x) = (A \circ L)f(x),\ f \in C[0,1].$$

Now Corollary 4.3.2 implies

Proposition 4.5.1. *With the above notation we have*

$$\begin{aligned}|(A \circ L)f(x) - Lf(x)| &\le \widetilde{\omega}\left(f; c_L A(|t-x|, x)\right)\\ &\le \widetilde{\omega}\left(f; c_L (A((t-x)^2, x))^{\frac{1}{2}}\right),\end{aligned}$$

for all $f \in C[0,1]$ and $x \in [0,1]$.

Application 4.5.2. Let $L = \overline{\mathbb{B}}_n$, where (see [76], [5] and the references therein)

$$\overline{\mathbb{B}}_n f(x) := \begin{cases} f(0) & \text{, if } x = 0,\\ f(1) & \text{, if } x = 1,\\ \frac{\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1} f(t)\mathrm{d}t}{B(nx, n(1-x))} & \text{, if } 0 < x < 1,\end{cases}$$

for all $f \in C[0,1]$, $x \in [0,1]$. Here $B(\cdot, \star)$ denotes the Beta function. A definition of these Beta operators $\overline{\mathbb{B}}_n$ can be found in Lupaş' thesis [75] (see p. 63). Then $\overline{\mathbb{B}}_n e_0 = e_0$.

In [5] it was proven using probabilistic methods that $\overline{\mathbb{B}}_n f$ is increasing whenever f is increasing.

Remark 4.5.3. This shape-preserving property can be proved, as in ([11], Ex.3.1), using analytical tools involving total positivity, not only for $\overline{\mathbb{B}}_n$ but also for the Beta operators $\mathcal{B}_n^{\alpha,\beta}$ which will be described in Application 4.5.5.

Let $g \in Lip[0,1]$. Then $|g|_{Lip} e_0 \pm g$ are increasing functions, so that $|g|_{Lip} e_0 \pm \overline{\mathbb{B}}_n g$ are also increasing. It follows that $\left|\overline{\mathbb{B}}_n g\right|_{Lip} \le |g|_{Lip}$, and so $c_{\overline{\mathbb{B}}_n} = 1$.

Take $A = B_n$, the classical Bernstein operator. Then $A \circ L = U_n$, the genuine Bernstein-Durrmeyer operator (see, e.g., [48]). From Proposition 4.5.1 we get

$$\left|U_n f(x) - \overline{\mathbb{B}}_n f(x)\right| \le \widetilde{\omega}\left(f; \left(\frac{x(1-x)}{n}\right)^{\frac{1}{2}}\right).$$

Application 4.5.4. Let $L = B_n$ and $A = \overline{\mathbb{B}}_n$. Then $A \circ L = S_n$ is a Stancu operator investigated in [76]. We infer that

$$|S_n f(x) - B_n f(x)| \leq \widetilde{\omega}\left(f; \left(\frac{x(1-x)}{n+1}\right)^{\frac{1}{2}}\right).$$

Application 4.5.5. For $\alpha, \beta > -1$, let $\mathcal{B}_n^{\alpha,\beta} : C[0,1] \to C[0,1]$ be defined by

$$\mathcal{B}_n^{\alpha,\beta} f(x) := \left(\int_0^1 t^{nx+\alpha}(1-t)^{n(1-x)+\beta} f(t)\mathrm{d}t\right) \Big/ \left(\int_0^1 t^{nx+\alpha}(1-t)^{n(1-x)+\beta}\mathrm{d}t\right).$$

As in Application 4.5.2 (see also Remark 4.5.3), it can be proven that the corresponding parameter $c_{\mathcal{B}_n^{\alpha,\beta}}$ is $\frac{n}{n+\alpha+\beta+2}$.

If we take $L = \mathcal{B}_n^{\alpha,\beta}$ and $A = B_n$, then $A \circ L = M_n^{\alpha,\beta}$, the Bernstein-Durrmeyer operator with Jacobi weights discussed in Application 4.4.3. From Proposition 4.5.1 we obtain

$$\left|M_n^{\alpha,\beta} f(x) - \mathcal{B}_n^{\alpha,\beta} f(x)\right| \leq \widetilde{\omega}\left(f; \frac{n}{n+\alpha+\beta+2}\left(\frac{x(1-x)}{n}\right)^{\frac{1}{2}}\right).$$

In particular, we can see what happens when $\alpha \to \infty$ and/or $\beta \to \infty$.

Application 4.5.6. Let $L = B_{n+1}$ and $A = B_n$. Then $A \circ L = D_n$, an operator which was investigated in [48]. In this case we have

$$|D_n f(x) - B_{n+1} f(x)| \leq \widetilde{\omega}\left(f; \left(\frac{x(1-x)}{n}\right)^{\frac{1}{2}}\right).$$

Remark 4.5.7. Other kind of results concerning differences of positive linear operators can be found in [48], [51], [52], [54] and the references therein.

5 Bivariate Ostrowski Inequalities

5.1 Auxiliary and historical results

Multivariate Ostrowski inequalities and generalizations of such results were considered by many mathematicians, such as G.A. Anastassiou, S.S. Dragomir, B.G. Pachpatte and many others. However, we only consider the bivariate case in the sequel and apply our results to some positive linear operators.

5.2 Bivariate positive linear operators

5.2.1 Bivariate Bernstein-Stancu operators

For $0 < \beta < \gamma$, consider the operators $S_n^{<0,\beta,\gamma>} : C([0,1]^2) \to C([0,1]^2)$ given by

$$S_n^{<0,\beta,\gamma>}(f;x,y) := \sum_{i,j=0}^{n} b_{n,i}(x) b_{n,j}(y) f\left(\frac{i+\beta}{n+\gamma}, \frac{j+\beta}{n+\gamma}\right).$$

We take the metric

$$d((x_1,y_1),(x_2,y_2)) := |x_1 - x_2| + |y_1 - y_2|$$

and consider the space

$$Lip([0,1]^2) := \left\{ f \in C([0,1]^2) : |f|_{Lip} := \sup_{(x_1,y_1)\neq(x_2,y_2)} \frac{|f(x_1,y_1) - f(x_2,y_2)|}{d((x_1,y_1),(x_2,y_2))} < \infty \right\}.$$

Let $f \in Lip([0,1]^2)$. We need to estimate the difference

$$\begin{aligned}
&\left| S_n^{<0,\beta,\gamma>} f(x_1,y_1) - S_n^{<0,\beta,\gamma>} f(x_2,y_2) \right| \\
&= \left| (x_1 - x_2) \frac{\partial (S_n^{<0,\beta,\gamma>} f)}{\partial x}(u,v) + (y_1 - y_2) \frac{\partial (S_n^{<0,\beta,\gamma>} f)}{\partial y}(u,v) \right|,
\end{aligned}$$

where $u = (1-t)x_1 + tx_2$, $v = (1-t)y_1 + ty_2$, for some $t \in (0,1)$.

We first estimate from above the following quantity.

$$\begin{aligned}
&\left| \frac{\partial (S_n^{<0,\beta,\gamma>} f)}{\partial x}(u,v) \right| = \left| \sum_{j=0}^{n} b_{n,j}(v) \sum_{i=0}^{n} b'_{n,i}(u) f\left(\frac{i+\beta}{n+\gamma}, \frac{j+\beta}{n+\gamma}\right) \right| \\
&= \left| \sum_{j=0}^{n} b_{n,j}(v) \cdot n \sum_{k=0}^{n-1} b_{n-1,k}(u) \left(f\left(\frac{k+1+\beta}{n+\gamma}, \frac{j+\beta}{n+\gamma}\right) - f\left(\frac{k+\beta}{n+\gamma}, \frac{j+\beta}{n+\gamma}\right) \right) \right| \qquad (5.2.1) \\
&\leq \sum_{j=0}^{n} b_{n,j}(v) \cdot n \sum_{k=0}^{n-1} b_{n-1,k}(u) \, |f|_{Lip} \cdot \frac{1}{n+\gamma} = \frac{n}{n+\gamma} \, |f|_{Lip}.
\end{aligned}$$

For the second equality from above (5.2.1), we recall that for the first derivative of the Bernstein operator, we have

$$B_n^{'}(f;x) = n \sum_{h=0}^{n-1} \binom{n-1}{h} \left[f\left(\frac{h+1}{n}\right) - f\left(\frac{h}{n}\right) \right] x^h (1-x)^{n-1-h}.$$

The same holds for the first derivative of $S_n^{<0,\beta,\gamma>} f$:

$$\begin{aligned}
&\sum_{i=0}^{n} b_{n,i}^{'}(u) f\left(\frac{i+\beta}{n+\gamma}, \frac{j+\beta}{n+\gamma}\right) \\
&= n \sum_{k=0}^{n-1} \binom{n-1}{k} \left[f\left(\frac{k+\beta+1}{n+\gamma}, \frac{j+\beta}{n+\gamma}\right) - f\left(\frac{k+\beta}{n+\gamma}, \frac{j+\beta}{n+\gamma}\right) \right] \cdot u^k (1-u)^{n-1-k} \\
&= n \sum_{k=0}^{n-1} b_{n-1,k}(u) \left[f\left(\frac{k+\beta+1}{n+\gamma}, \frac{j+\beta}{n+\gamma}\right) - f\left(\frac{k+\beta}{n+\gamma}, \frac{j+\beta}{n+\gamma}\right) \right].
\end{aligned}$$

Analogously, we get

$$\left| \frac{\partial (S_n^{<0,\beta,\gamma>} f)}{\partial y}(u,v) \right| \leq \frac{n}{n+\gamma} |f|_{Lip} .$$

Now the desired difference can be estimated by

$$\left| S_n^{<0,\beta,\gamma>} f(x_1,y_1) - S_n^{<0,\beta,\gamma>} f(x_2,y_2) \right| \leq \frac{n}{n+\gamma} |f|_{Lip} \left(|x_1 - x_2| + |y_1 - y_2| \right),$$

from where we can see that $S_n^{<0,\beta,\gamma>} f \in Lip([0,1]^2)$ and $\left| S_n^{<0,\beta,\gamma>} f \right|_{Lip} \leq \frac{n}{n+\gamma} |f|_{Lip}$.

We need to determine an invariant measure of the form

$$\mu = \sum_{k,l=0}^{n} c_{kl} \delta_{\left(\frac{k+\beta}{n+\gamma}, \frac{l+\beta}{n+\gamma}\right)},$$

for $c_{kl} \in \mathbb{R}$ and satisfying $\sum_{k,l} c_{kl} = 1$. We want to have that $\mu(S_n^{<0,\beta,\gamma>} f) = \mu(f)$, for all $f \in C([0,1]^2)$, i.e.,

$$\sum_{k,l=0}^{n} c_{kl} (S_n^{<0,\beta,\gamma>} f) \left(\frac{k+\beta}{n+\gamma}, \frac{l+\beta}{n+\gamma}\right) = \sum_{k,l=0}^{n} c_{kl} f \left(\frac{k+\beta}{n+\gamma}, \frac{l+\beta}{n+\gamma}\right),$$

for all f. It holds

$$\begin{aligned}
&\sum_{k,l=0}^{n} c_{kl} \sum_{i,j=0}^{n} b_{n,i}\left(\frac{k+\beta}{n+\gamma}\right) b_{n,j}\left(\frac{l+\beta}{n+\gamma}\right) f\left(\frac{i+\beta}{n+\gamma}, \frac{j+\beta}{n+\gamma}\right) \\
&= \sum_{i,j=0}^{n} c_{ij} f\left(\frac{i+\beta}{n+\gamma}, \frac{j+\beta}{n+\gamma}\right) \Rightarrow \\
&\sum_{i,j=0}^{n} f\left(\frac{i+\beta}{n+\gamma}, \frac{j+\beta}{n+\gamma}\right) \sum_{k,l=0}^{n} c_{kl} b_{n,i}\left(\frac{k+\beta}{n+\gamma}\right) b_{n,j}\left(\frac{l+\beta}{n+\gamma}\right) \\
&= \sum_{i,j=0}^{n} c_{ij} f\left(\frac{i+\beta}{n+\gamma}, \frac{j+\beta}{n+\gamma}\right) \Rightarrow \\
&\sum_{i,j=0}^{n} f\left(\frac{i+\beta}{n+\gamma}, \frac{j+\beta}{n+\gamma}\right) \left[\sum_{k,l=0}^{n} c_{kl} b_{n,i}\left(\frac{k+\beta}{n+\gamma}\right) b_{n,j}\left(\frac{l+\beta}{n+\gamma}\right) - c_{ij} \right] = 0, \ \forall f,
\end{aligned}$$

which implies that

$$\sum_{k,l=0}^{n} b_{n,i}\left(\frac{k+\beta}{n+\gamma}\right) b_{n,j}\left(\frac{l+\beta}{n+\gamma}\right) c_{kl} = c_{ij} \tag{5.2.2}$$

holds, for all $i, j = 0, \ldots, n$.

Consider the $(n+1)^2 \times (n+1)^2$ matrix $B = \left(b_{n,i}\left(\frac{k+\beta}{n+\gamma}\right) b_{n,j}\left(\frac{l+\beta}{n+\gamma}\right)\right)$, where the row index (i, j) and the column index (k, l) take the values $(0,0), (0,1), \ldots, (0,n)$, $(1,0), (1,1), \ldots, (1,n), \ldots, (n,0), (n,1), \ldots, (n,n)$.

All the entries of B are > 0 and the sum of the entries on each column is 1. Equation (5.2.2) shows that the vector

$$\overline{w} = (c_{00}, c_{01}, \ldots, c_{0n}, c_{10}, c_{11}, \ldots, c_{1n}, \ldots, c_{n0}, c_{n1}, \ldots, c_{nn})^t$$

is an eigenvector of B corresponding to the eigenvalue 1, i.e., $B\overline{w} = \overline{w}$. It is known that the system

$$\begin{cases} B\overline{w} = \overline{w} \\ \sum_{k,l=0}^{n} c_{kl} = 1 \end{cases}$$

has a unique solution and it has the property $c_{kl} > 0$, for all $k, l = 0, \ldots, n$. Thus we can conclude that the invariant measure μ exists and is unique.

5.2.2 Bivariate Bernstein-Durrmeyer operators with Jacobi weights

We will present a similar approach to the one used for the Bernstein-Stancu operators. However, first we define the Bernstein-Durrmeyer operator with Jacobi weights $M_n^{\alpha,\beta} : C([0,1]^2) \to C([0,1]^2)$. This is given by

$$M_n^{\alpha,\beta} f(x,y) := \sum_{i,j=0}^{n} b_{n,i}(x) \cdot b_{n,j}(y) \cdot a_{n;i,j}(f; x, y),$$

where

$$a_{n;i,j}(f; x, y) := \frac{\iint\limits_{[0,1]^2} x^{i+\alpha}(1-x)^{n-i+\beta} y^{j+\alpha}(1-y)^{n-j+\beta} f(x,y)\mathrm{d}x\mathrm{d}y}{\iint\limits_{[0,1]^2} x^{i+\alpha}(1-x)^{n-i+\beta} y^{j+\alpha}(1-y)^{n-j+\beta}\mathrm{d}x\mathrm{d}y}$$

and $x, y \in [0,1]^2$.

We take the same metric like before, i.e.,

$$d(x,y) = d((x_1,x_2),(y_1,y_2)) := |x_1 - x_2| + |y_1 - y_2|,$$

and we again consider the space of functions

$$Lip([0,1]^2) := \left\{ f \in C([0,1]^2) \,|\, |f|_{Lip} := \sup_{(x_1,y_1)\neq(x_2,y_2)} \frac{|f(x_1,y_1) - f(x_2,y_2)|}{d((x_1,y_1),(x_2,y_2))} < \infty \right\}.$$

Let $f \in Lip([0,1]^2)$ and estimate the following difference:

$$\begin{aligned}
&\left| M_n^{\alpha,\beta}(f;x_1,y_1) - M_n^{\alpha,\beta}(f;x_2,y_2) \right| \\
&= \left| (x_1 - x_2) \cdot \frac{\partial(M_n^{\alpha,\beta} f)}{\partial x}(u,v) + (y_1 - y_2) \frac{\partial(M_n^{\alpha,\beta} f)}{\partial y}(u,v) \right|,
\end{aligned}$$

for $u = (1-t)x_1 + tx_2$, $v = (1-t)y_1 + ty_2$, with some $t \in (0,1)$.

We have

$$\begin{aligned}
&\left| \frac{\partial(M_n^{\alpha,\beta} f)}{\partial x}(u,v) \right| = \left| \sum_{j=0}^{n} b_{n,j}(v) \cdot \sum_{i=0}^{n} b_{n,i}^{'}(u) \cdot a_{n;i,j}(f;u,v) \right| \\
&= \left| \sum_{j=0}^{n} b_{n,j}(v) \cdot n \cdot \sum_{i=0}^{n-1} b_{n-1,i}(u) \cdot \left[a_{n;i+1,j}(f;u,v) - a_{n;i,j}(f;u,v) \right] \right| \\
&\leq \sum_{j=0}^{n} b_{n,j}(v) \cdot n \cdot \sum_{i=0}^{n-1} b_{n-1,i}(u) \cdot \left| a_{n;i+1,j}(f;u,v) - a_{n;i,j}(f;u,v) \right|
\end{aligned}$$

and we want to estimate the quantity in the absolute value on the right hand-side of the inequality.

For $f \in Lip([0,1]^2)$, $i = \{0, \ldots, n-1\}$, we introduce the function

$$F_1(x,y) := x^{i+\alpha+1} \cdot (1-x)^{n-i+\beta} \cdot y^{j+\alpha} \cdot (1-y)^{n-j+\beta},$$

for $0 \leq x, y \leq 1$. The derivative of this function with respect to x is

$$\begin{aligned}
&F_{1,x}'(x,y) := -y^{j+\alpha}(1-y)^{n-j+\beta} \\
&\cdot \left[(n-i+\beta) x^{i+\alpha+1} (1-x)^{n-i+\beta-1} - (i+\alpha+1) x^{i+\alpha} (1-x)^{n-i+\beta} \right].
\end{aligned}$$

Then we obtain

$$
\begin{aligned}
&a_{n;i+1,j}(f) - a_{n;i,j}(f)\\
&=\frac{\iint\limits_{[0,1]^2} x^{i+\alpha+1}(1-x)^{n-i+\beta-1}y^{j+\alpha}(1-y)^{n-j+\beta}f(x,y)\mathrm{d}x\mathrm{d}y}{\iint\limits_{[0,1]^2} x^{i+\alpha+1}(1-x)^{n-i+\beta-1}y^{j+\alpha}(1-y)^{n-j+\beta}\mathrm{d}x\mathrm{d}y}-\\
&-\frac{\iint\limits_{[0,1]^2} x^{i+\alpha}(1-x)^{n-i+\beta}y^{j+\alpha}(1-y)^{n-j+\beta}f(x,y)\mathrm{d}x\mathrm{d}y}{\iint\limits_{[0,1]^2} x^{i+\alpha}(1-x)^{n-i+\beta}y^{j+\alpha}(1-y)^{n-j+\beta}\mathrm{d}x\mathrm{d}y}\\
&=\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(i+\alpha+2)\Gamma(n-i+\beta)}\cdot\frac{\iint\limits_{[0,1]^2} x^{i+\alpha+1}(1-x)^{n-i+\beta-1}y^{j+\alpha}(1-y)^{n-j+\beta}f(x,y)\mathrm{d}x\mathrm{d}y}{\int_0^1 y^{j+\alpha}(1-y)^{n-j+\beta}\mathrm{d}y}\\
&-\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(i+\alpha+1)\Gamma(n-i+\beta+1)}\cdot\frac{\iint\limits_{[0,1]^2} x^{i+\alpha}(1-x)^{n-i+\beta}y^{j+\alpha}(1-y)^{n-j+\beta}f(x,y)\mathrm{d}x\mathrm{d}y}{\int_0^1 y^{j+\alpha}(1-y)^{n-j+\beta}\mathrm{d}y}\\
&=\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(i+\alpha+2)\Gamma(n-i+\beta)}\cdot\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(j+\alpha+1)\Gamma(n-j+\beta+1)}\cdot\\
&\cdot\iint\limits_{[0,1]^2} x^{i+\alpha+1}(1-x)^{n-i+\beta-1}y^{j+\alpha}(1-y)^{n-j+\beta}f(x,y)\mathrm{d}x\mathrm{d}y\\
&-\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(i+\alpha+1)\Gamma(n-i+\beta+1)}\cdot\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(j+\alpha+1)\Gamma(n-j+\beta+1)}\cdot\\
&\cdot\iint\limits_{[0,1]^2} x^{i+\alpha}(1-x)^{n-i+\beta}y^{j+\alpha}(1-y)^{n-j+\beta}f(x,y)\mathrm{d}x\mathrm{d}y\\
&=\frac{\Gamma(n+\alpha+\beta+2)^2}{\Gamma(i+\alpha+2)\Gamma(n-i+\beta+1)\Gamma(j+\alpha+1)\Gamma(n-j+\beta+1)}\cdot\\
&\cdot\{C-D\},
\end{aligned}
$$

where

$$C := \iint\limits_{[0,1]^2} (n-i+\beta)x^{i+\alpha+1}(1-x)^{n-i+\beta-1}y^{j+\alpha}(1-y)^{n-j+\beta}f(x,y)\mathrm{d}x\mathrm{d}y$$

and

$$D := \iint\limits_{[0,1]^2} (i+\alpha+1)x^{i+\alpha}(1-x)^{n-i+\beta}y^{j+\alpha}(1-y)^{n-j+\beta}f(x,y)\mathrm{d}x\mathrm{d}y\}.$$

The above quantity $C-D$ can be written as follows:

$$
\begin{aligned}
&C-D := \int_0^1\left(y^{j+\alpha}(1-y)^{n-j+\beta}\int_0^1 (n-i+\beta)x^{i+\alpha+1}(1-x)^{n-i+\beta-1}f(x,y)\mathrm{d}x\right)\mathrm{d}y\\
&-\int_0^1\left(y^{j+\alpha}(1-y)^{n-j+\beta}\int_0^1 (i+\alpha+1)x^{i+\alpha}(1-x)^{n-i+\beta}f(x,y)\mathrm{d}x\right)\mathrm{d}y\\
&=-\int_0^1\int_0^1 F'_{1,x}(x,y)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

So it holds

$$
\begin{aligned}
&a_{n;i+1,j}(f) - a_{n;i,j}(f) \\
&= -\frac{\Gamma(n+\alpha+\beta+2)^2}{\Gamma(i+\alpha+2)\Gamma(n-i+\beta+1)\Gamma(j+\alpha+1)\Gamma(n-j+\beta+1)} \cdot \int_0^1 \left[\int_0^1 F'_{1,x}(x,y)\mathrm{d}x\right] \mathrm{d}y \\
&= -\frac{\Gamma(n+\alpha+\beta+2)^2}{\Gamma(i+\alpha+2)\Gamma(n-i+\beta+1)\Gamma(j+\alpha+1)\Gamma(n-j+\beta+1)} \cdot \int_0^1 \left(\int_0^1 f(\cdot,y) dF_1(\cdot,y)\right) \mathrm{d}y \\
&= -\frac{\Gamma(n+\alpha+\beta+2)^2}{\Gamma(i+\alpha+2)\Gamma(n-i+\beta+1)\Gamma(j+\alpha+1)\Gamma(n-j+\beta+1)} \\
&\cdot \int_0^1 \left[f(\cdot,y)F_1(\cdot,y)|_0^1 + \int_0^1 F_1(\cdot,y) df(\cdot,y)\right] \mathrm{d}y \\
&= -\frac{\Gamma(n+\alpha+\beta+2)^2}{\Gamma(i+\alpha+2)\Gamma(n-i+\beta+1)\Gamma(j+\alpha+1)\Gamma(n-j+\beta+1)} \\
&\cdot \int_0^1 \left(\int_0^1 F_1(\cdot,y) df(\cdot,y)\right) \mathrm{d}y
\end{aligned}
$$

Now we consider the absolute value of the above quantity and we have

$$
\begin{aligned}
&\left|a_{n;i+1,j}(f) - a_{n;i,j}(f)\right| \\
&\leq \frac{\Gamma(n+\alpha+\beta+2)^2}{\Gamma(i+\alpha+2)\Gamma(n-i+\beta+1)\Gamma(j+\alpha+1)\Gamma(n-j+\beta+1)} \cdot \int_0^1 \left|\int_0^1 F_1(\cdot,y)\mathrm{d}f(\cdot,y)\right| \mathrm{d}y \\
&= \frac{\Gamma(n+\alpha+\beta+2)^2}{\Gamma(i+\alpha+2)\Gamma(n-i+\beta+1)\Gamma(j+\alpha+1)\Gamma(n-j+\beta+1)} \cdot \int_0^1 \left(\int_0^1 F_1(x,y)\mathrm{d}x\mathrm{d}y\right) |f|_{Lip} \\
&= \frac{\Gamma(n+\alpha+\beta+2)^2}{\Gamma(i+\alpha+2)\Gamma(n-i+\beta+1)\Gamma(j+\alpha+1)\Gamma(n-j+\beta+1)} \cdot |f|_{Lip} \\
&\cdot \iint\limits_{[0,1]^2} x^{i+\alpha+1}(1-x)^{n-i+\beta} y^{j+\alpha}(1-y)^{n-j+\beta} \mathrm{d}x\mathrm{d}y \\
&= \frac{\Gamma(n+\alpha+\beta+2)^2}{\Gamma(i+\alpha+2)\Gamma(n-i+\beta+1)\Gamma(j+\alpha+1)\Gamma(n-j+\beta+1)} \cdot |f|_{Lip} \cdot \\
&\cdot \frac{\Gamma(j+\alpha+1)\Gamma(n-j+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \cdot \frac{\Gamma(i+\alpha+2)\Gamma(n-i+\beta+1)}{\Gamma(n+\alpha+\beta+3)} \\
&= \frac{1}{n+\alpha+\beta+2} \cdot |f|_{Lip} .
\end{aligned}
$$

By gathering all the information we have, we get

$$
\begin{aligned}
&\left|\frac{\partial(M_n^{\alpha,\beta}(f))}{\partial x}(u,v)\right| \\
&\leq \sum_{j=0}^{n} b_{n,j}(v) \cdot n \sum_{i=0}^{n-1} b_{n-1,i}(u) \cdot \frac{1}{n+\alpha+\beta+2} |f|_{Lip} \\
&\leq \frac{n}{n+\alpha+\beta+2} |f|_{Lip} .
\end{aligned}
$$

In the same way, we obtain the other partial derivative with respect to y, as seen in the sequel.

$$\begin{aligned}
&\left|\frac{\partial(M_n^{\alpha,\beta}(f))}{\partial y}(u,v)\right| \\
&= \left|\sum_{i=0}^{n} b_{n,i}(u)\cdot\sum_{j=0}^{n} b'_{n,j}(v)\cdot a_{n;i,j}(f;u,v)\right| \\
&= \left|\sum_{i=0}^{n} b_{n,i}(u)\cdot n\sum_{j=0}^{n-1} b_{n-1,j}(u)\cdot\left[a_{n;i,j+1}(f)-a_{n;i,j}(f)\right]\right| \\
&\le \sum_{i=0}^{n} b_{n,i}(u)\cdot n\sum_{j=0}^{n-1} b_{n-1,j}(v)\cdot\left|a_{n;i,j+1}(f)-a_{n;i,j}(f)\right|.
\end{aligned}$$

We estimate the absolute value from the last line above using the same method as before. This time, for $f \in Lip([0,1]^2)$ and $j \in \{0,\dots,n-1\}$, we introduce the function

$$F_2(x,y) := x^{i+\alpha}(1-x)^{n-i+\beta}y^{j+\alpha+1}(1-y)^{n-j+\beta},\ 0 \le x,y \le 1,$$

and its derivative with respect to y is

$$\begin{aligned}
&F'_{2,y}(x,y) \\
&= -x^{i+\alpha}(1-x)^{n-i+\beta}\left[(n-j+\beta)y^{j+\alpha+1}(1-y)^{n-j+\beta-1}-(j+\alpha+1)y^{j+\alpha}(1-y)^{n-j+\beta}\right].
\end{aligned}$$

Then after some calculations, proceeding just like before, we obtain a similar result:

$$\left|\frac{\partial(M_n^{\alpha,\beta}(f))}{\partial y}(u,v)\right| \le \frac{n}{n+\alpha+\beta+2}\cdot|f|_{Lip}.$$

From the two estimates from above, concerning the partial derivatives of our operator, we have

$$\begin{aligned}
&\left|M_n^{\alpha,\beta}f(x_1,y_1)-M_n^{\alpha,\beta}f(x_2,y_2)\right| \\
&\le \frac{n}{n+\alpha+\beta+2}\,|f|_{Lip}\left(|x_1-x_2|+|y_1-y_2|\right).
\end{aligned}$$

From this it follows that $M_n^{\alpha,\beta}f \in Lip([0,1]^2)$ and

$$\left|M_n^{\alpha,\beta}f\right|_{Lip} \le \frac{n}{n+\alpha+\beta+2}\,|f|_{Lip}.$$

5.3 Main results

5.3.1 An Ostrowski inequality for the bivariate Bernstein-Stancu operator

Let $f \in Lip([0,1]^2)$. We have the estimate

$$\begin{aligned}
\left| f(x,y) - \iint\limits_{[0,1]^2} f d\mu \right| &= \left| f(x,y) - \sum_{k,l=0}^{n} c_{kl} f\left(\frac{k+\beta}{n+\gamma}, \frac{l+\beta}{n+\gamma}\right) \right| \\
&= \left| \sum_{k,l=0}^{n} c_{kl} \left(f(x,y) - f\left(\frac{k+\beta}{n+\gamma}, \frac{l+\beta}{n+\gamma}\right)\right) \right| \\
&\leq \sum_{k,l=0}^{n} c_{kl} \left| f(x,y) - f\left(\frac{k+\beta}{n+\gamma}, \frac{l+\beta}{n+\gamma}\right) \right| \\
&\leq \sum_{k,l=0}^{n} c_{kl} \, |f|_{Lip} \left(\left| x - \frac{k+\beta}{n+\gamma} \right| + \left| y - \frac{l+\beta}{n+\gamma} \right| \right) \\
&= |f|_{Lip} \underbrace{\sum_{k,l=0}^{n} c_{kl} \left(\left| x - \frac{k+\beta}{n+\gamma} \right| + \left| y - \frac{l+\beta}{n+\gamma} \right| \right)}_{=:w(x,y)},
\end{aligned}$$

so the Ostrowski inequality becomes

$$\left| f(x,y) - \iint\limits_{[0,1]^2} f d\mu \right| \leq |f|_{Lip} \cdot w(x,y)$$

Now considering the over-iterates of the bivariate Bernstein-Stancu operator and applying them to the above inequality, we have the following result:

$$\left| \left(S_n^{<0,\beta,\gamma>}\right)^m f(x,y) - \iint\limits_{[0,1]^2} \left(S_n^{<0,\beta,\gamma>}\right)^m f d\mu \right| \leq \left| \left(S_n^{<0,\beta,\gamma>}\right)^m f \right|_{Lip} \cdot w(x,y),$$

which, because of our invariant measure μ, is the same as writing

$$\left| \left(S_n^{<0,\beta,\gamma>}\right)^m f(x,y) - \iint\limits_{[0,1]^2} f d\mu \right| \leq \left| \left(S_n^{<0,\beta,\gamma>}\right)^m f \right|_{Lip} \cdot w(x,y).$$

From a result we obtained before we can easily see that

$$\left| \left(S_n^{<0,\beta,\gamma>}\right)^m f(x,y) - \iint\limits_{[0,1]^2} f d\mu \right| \leq \left(\frac{n}{n+\gamma}\right)^m |f|_{Lip} \cdot w(x,y)$$

holds, so we get

$$\lim_{m\to\infty} \left(S_n^{<0,\beta,\gamma>}\right)^m f = \left(\iint\limits_{[0,1]^2} f d\mu \right) \mathbb{1},$$

for all $f \in Lip([0,1]^2)$ and all $n \geq 1$. Since $Lip([0,1]^2)$ is dense in $C([0,1]^2)$ with respect to the uniform norm, we have

$$\lim_{m\to\infty} \left(S_n^{<0,\beta,\gamma>}\right)^m f = \left(\iint\limits_{[0,1]^2} f d\mu\right) \mathbb{1},$$

for all $f \in C([0,1]^2)$ and all $n \geq 1$.

5.3.2 An Ostrowski inequality for the bivariate Bernstein-Durrmeyer operators with Jacobi weights

Let $f \in Lip([0,1]^2)$. Then we get the Ostrowski inequality for the bivariate Bernstein-Durrmeyer operator with Jacobi weights as follows.

We have the estimate

$$\begin{aligned}
\left| f(x,y) - \iint\limits_{[0,1]^2} f(s,t)\mathrm{d}\mu(s,t) \right| &= \left| \iint\limits_{[0,1]^2} [f(x,y) - f(s,t)]\,\mathrm{d}\mu(s,t) \right| \\
&\leq \iint\limits_{[0,1]^2} |f(x,y) - f(s,t)|\,\mathrm{d}\mu(s,t) \\
&\leq \iint\limits_{[0,1]^2} |f|_{Lip}\,(|x-s| + |y-t|)\,\mathrm{d}\mu(s,t) \\
&= |f|_{Lip} \cdot w(x,y),
\end{aligned}$$

where

$$w(x,y) := \frac{\iint\limits_{[0,1]^2} (|x-s| + |y-t|)\, s^\alpha (1-s)^\beta t^\alpha (1-t)^\beta \mathrm{d}s\mathrm{d}t}{\iint\limits_{[0,1]^2} s^\alpha (1-s)^\beta t^\alpha (1-t)^\beta \mathrm{d}s\mathrm{d}t}.$$

For the over-iterates of the bivariate Bernstein-Durrmeyer operator with Jacobi weights, the following inequalities hold,

$$\begin{aligned}
\left| (M_n^{\alpha,\beta})^m f(x,y) - \iint\limits_{[0,1]^2} f\mathrm{d}\mu \right| &\leq \left| (M_n^{\alpha,\beta})^m f \right|_{Lip} \cdot w(x,y) \\
&\leq \left(\frac{n}{n+\alpha+\beta+2}\right)^m \cdot |f|_{Lip} \cdot w(x,y),
\end{aligned}$$

so we get

$$\lim_{m\to\infty} (M_n^{\alpha,\beta})^m f = \left(\iint\limits_{[0,1]^2} f\mathrm{d}\mu\right) \mathbb{1},$$

for all $f \in Lip([0,1]^2)$ and all $n \geq 1$. Since $Lip([0,1]^2)$ is dense in $C([0,1]^2)$ with respect to the uniform norm, we have

$$\lim_{m\to\infty} (M_n^{\alpha,\beta})^m f = \left(\iint\limits_{[0,1]^2} f \, \mathrm{d}\mu \right) \mathbb{1},$$

for all $f \in C([0,1]^2)$ and all $n \geq 1$.

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